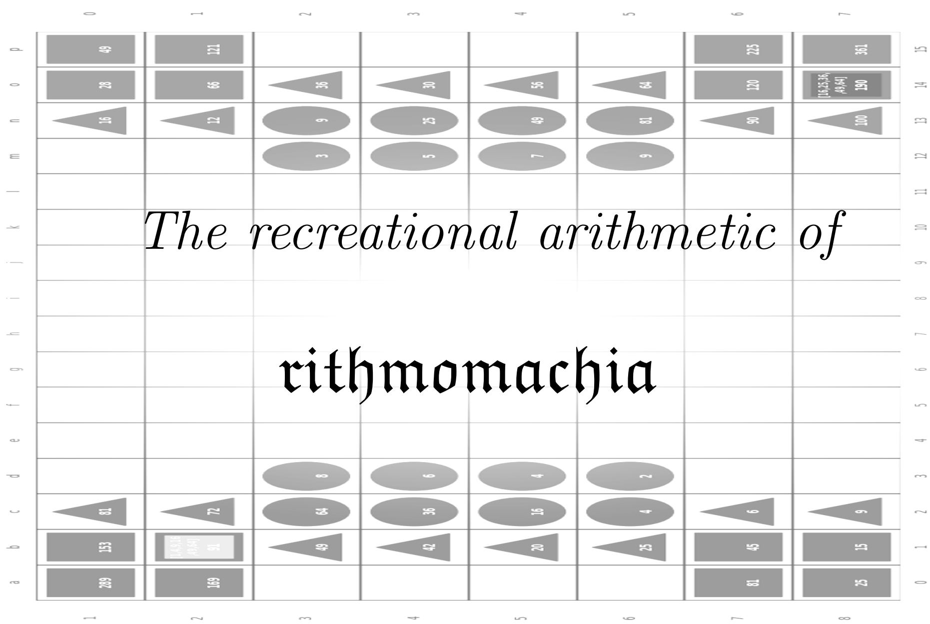

The recreational arithmetic of

rithmomachia

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Rithmomachia (or **rithmomachia** if you prefer:-) is a two-player board game invented in the middle ages. According to medieval mathematics scholars, using the arithmetically-based rules governing the rithmomachia pieces helps students with logical and spiritual reasoning, allowing them to attain a closer connection the cosmos. Indeed, some cathedral-based schools at the time *banned* chess which they believed could lead to vices (such as gambling!), but allowed their students to play rithmomachia.

In the sections below, the reader will find an introduction to the game, including a detailed description of the rules regarding

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the board and pieces. In particular, we describe the interesting way the pieces are set up initially, how they move, how they capture, and most importantly, how to win! Chess-like notation and easily readible board diagrams are used for examples and even a move-by-move presentation of two complete games.

Versions

Given the different rules among the different versions of rithmomachia, we narrow our focus here to the rules of the game described by Claude de Boissière in 1556 [Ri46].

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As you can tell from the table of contents, the organization is pretty simple. First, we talks about the history of the game, which involves an unusual combination of mathematics, religion and philosophy. After that background is sketched out, the game rules are covered, with examples. In the next section a complete game, using the notation introduced earlier, is given. Finally, there is a summary with some open questions. Fun exercises and problems (and cartoons:-) are scattered throughout.

Figure 1: Rithmomachia White triangles pieces in a line. (The graphics for the pieces were made using `SageMath` and Python.)

Let's start in the beginning, by explaining the fascinating Pythagorean philosophical origins of the game, the intruiguiging numerical relationships among its pieces, and how victory is seen as spiritual enlightenment.

1 The Philosopher's Game

It is known that the game goes back at least until the 11th century, but it may even go back further. As with chess, which has versions going back to 7th century Persia and India, this board game models a battle of sorts. Indeed, the *Rithmomachia* name itself means, roughly speaking, "battle with numbers."

Rithmomachia is indeed called the *Battle of Numbers* but it's also called the *Philosopher's Game*, and one medieval author even called it the *Pythagorean Game*, believing (perhaps incorrectly) that it dated back to the time of the ancient Greeks. What spiritual connections did religious medieval scholars believe this game had with the world as they saw it?

To monastic scholars, rithmomachia was more than a medieval math game — it was a ritual of contemplation. The Pythagorean philosophers during the middle ages believed

- numbers are sacred,
- harmony is divine, and
- through the process of playing rithmomachia, the player gains a better understanding of the metaphysics of the soul and the cosmos.

1.1 Pythagorean philosophy

This board game was popular in many parts of Europe from the 1100s to the 1600s [Mo01]. While a reasonable hypothesis, there is no historical evidence that the origin of the game goes back to Pythagoras (who lived around 600-500 BCE). What is known is that the Greek scholar Nicomachus, a Pythagorean who lived around 60-120 CE, produced a mathematical text that eventually inspired the rules for rithmomachia. Nicomachus is best known as the author of *Introduction to Arithmetic* and (the musical text) *Manual of Harmonics*. A semantical word of caution: in the time of Nicomachus, "arithmetic" had a different meaning that it does now. Then, arithmetic was a discipline of study – primarily a philosophical discipline concerned with the essence and underlying principles of numbers, such as classification and their mystical significance. Developing skill to perform practical calculations was a separate discipline of study. Nicomachus more often viewed numbers as objects having mystical properties rather than mathematical properties. His work was translated into Latin and popularized by the Roman scholar Boethius around 500 CE. More on this important character in our story can be found in §1.4 below.

While not spiritual in the sense of religious doctrine, rithmomachia is deeply embedded in the philosophical and mathematical worldview of the Middle Ages, particularly Neoplatonism and Boethian number theory. It was conceived as more than just a game; it was an educational tool designed to instill an understanding of cosmic order through numbers. Playing it well was seen as both an intellectual achievement and a spiritual practice.

The starting configuration on the 8×16 board (split between White and Black halves) reflects a moral and metaphysical battlefield:

- White (Even): Often associated with order, reason, and light — placed at the bottom, symbolizing the soul's ascent from material beginnings.
- Black (Odd): Associated with dynamism, passion, and will — beginning at the top, perhaps symbolizing the descent of spiritual forces into the material.

The symmetry and numerical elegance of the starting rows mimic a cosmic harmony — much like a music scale (which Pythagoreans also based on ratios). The game's goal, especially proper victories (by placing harmonically valued pieces in the enemy's half), mirrors the soul achieving harmony within the world.

Figure 2: rithmomachia pieces can get metaphysical. Thanks chatGPT:-).

1.2 Pieces and their spiritual significance

Rithmomachia features pieces with distinct shapes and numerical values that convey deep philosophical and spiritual meaning:

- The game uses four types of pieces: Circles, Triangles, Squares, and Pyramids. Each piece has a specific value assigned to it, with black and white forces having asymmetrical numerical values. This asymmetry was intentional and reflected medieval philosophical concepts about numerical harmony and cosmic order.

- The numerical values assigned to the pieces aren't arbitrary but represent mathematical proportions that help explain the fabric of the cosmos.

These numerical values and their relationships arose from the study of Boethian mathematical philosophy, which emphasized the natural harmony and perfection of number systems. According to the medieval interpretation, the world was created out of a knowledge of number and proportion (Boethius following Nicomachus, Pythagoras, and Plato), so these values are symbolic of spiritual laws that govern both the physical and metaphysical realms. The different shapes of pieces carried the range of Pythagorean symbolism:

- Triangles: Represent the trinity in Christian thought and the threefold nature of existence (mind/body/spirit). Their two space movement pattern symbolized movement between different stages. Triangles stand for the mind or reason, able to reach farther than the (circular) soul. These pieces represent developing intellectual capacity.

  > **Triangular numbers**
  >
  > ... count equally spaced dots inside an equilateral triangle (6, 10, 15, . . .). There is a simple algebraic formula for the nth triangular number: $n(n + 1)/2$. The Pythagoreans believed they represent growth, virtue, and human striving. They are associated with moral development.

- Squares: Moving three spaces in any orthogonal direction was reminder of the four elements (earth/wind/fire/water) and of the four seasons (summer/fall/winter/spring). Representing wisdom or even divine authority, their movement is the longest range — symbolizing fully realized power or divine insight.

Square numbers

... count equally spaced dots inside a square (4, 9, 16, ...). They represent stability and justice in the cosmos. Pythagoreans saw squares as the foundation of order.

- Pyramids: This is the most complicated and powerful piece. It also moves like a square, but one that contains several component subpieces of different values. Indeed, their total value is a sums of squares – White's pyramid has value

  $$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91,$$

  while Black's is

  $$4^2 + 5^2 + 6^2 + 7^2 + 8^2 = 16 + 25 + 36 + 49 + 64 = 190.$$

  It's a spiritually significant piece, representing ascension and the hierarchy of knowledge. This piece is symbolic of higher wisdom. The pyramid piece moves like a square and contains several component subpieces.

  - **White's Pyramid (Value 91)**: Composed of six white subpieces with values $1^2, 2^2, 3^2, 4^2, 5^2, 6^2$ (i.e., 1, 4, 9, 16, 25, 36).
  - **Black's Pyramid (Value 190)**: Composed of five black subpieces with values $4^2, 5^2, 6^2, 7^2, 8^2$ (i.e., 16, 25, 36, 49, 64).

Pyramid numbers

... count equally spaced dots inside a square-based pyramid (1, 5, 14, ...). There is a formula for the nth such number: $Pyr_n = n(n+1)(2n+1)/6$. The first 9 pyramid numbers are:

$$1, 5, 14, 30, 55, 91, 140, 204.$$

Note that the 7th pyramid number is the value of the white pyramid. While the value 190 of the black pyramid is not a pyramid number, it is a difference of them: $190 = 204 - 14$. Imagine that this way: if you take the 8×8 pyramid and remove the 3×3 pyramid from its top, you get the black pyramid.
The pyramid numbers occured in the works of Nicomachus [Ni00].

- Circles (or rounds): Symbolized perfection and unity, representing the divine whole. Their limited movement (one space) reflected how fundamental principles operate within constrained natural laws. The one space they move is the smallest units — like the circle signifying the soul's first steps in understanding.

"Circular" numbers?

Unlike triangular numbers, square numbers, and pyramid numbers, the corresponding definition for the circle doesn't work the same. *If* we defined a "circular number" to be those numbers that count equally spaced dots inside a circle (1, 5, 13, ...) then it is *much* harder to find a formula for these. Indeed, the problem of estimating such numbers is called the *Gauss circle problem*, named after Carl Frederich Guass who first made progress in the early 1800s. Indeed, for such numbers, the term "circular number" is not traditionally used. Instead, it is called the "counting function for the Gauss circle problem." Much better:-)

The initial arrangement of pieces on the board reflected medieval concepts of universal order. This orderly arrangement mirrored the medieval concept of the universe as divinely ordered and mathematically harmonious. The numerical sequences used in the game followed mathematical progressions that were considered spiritually significant, such as arithmetic, geometric, and harmonic sequences. To the medieval students playing rithmomachia, these victorious patterns were not just game mechanics but reflections of divine proportions.

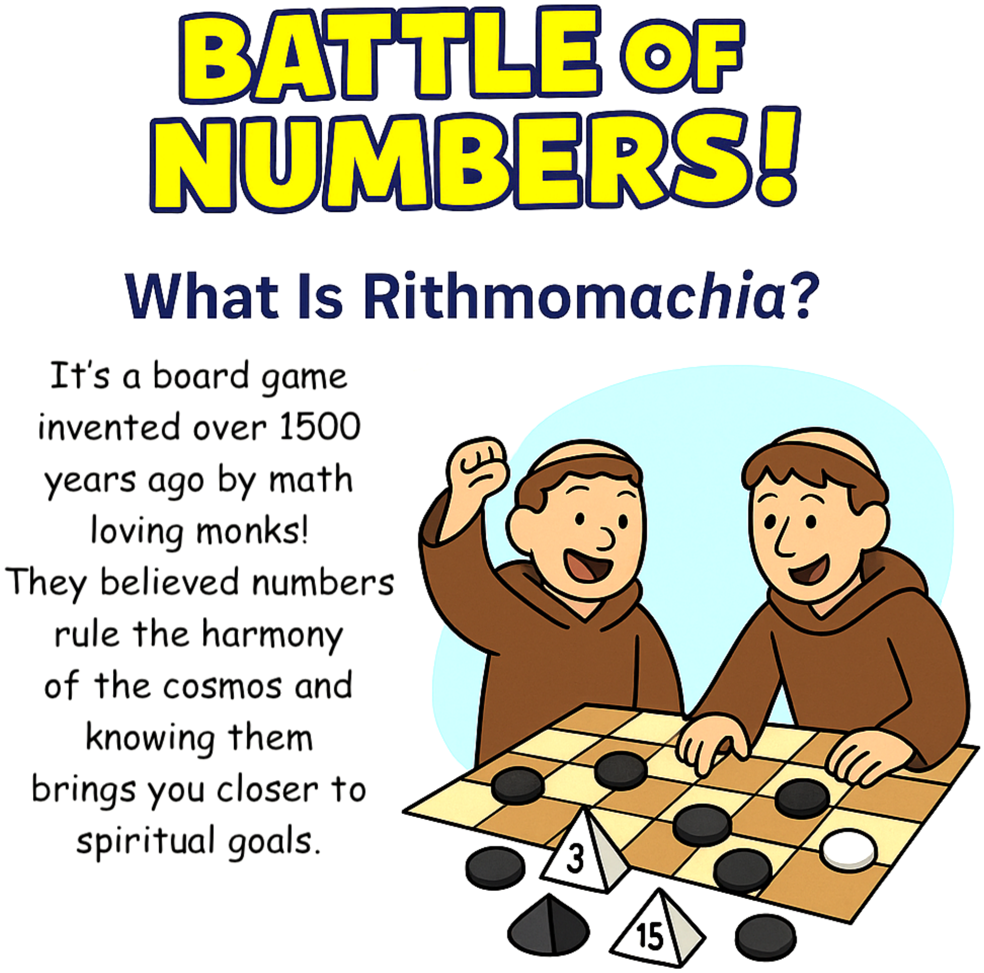
1.3 Victory as spiritual enlightenment

The various victory conditions in rithmomachia had spiritual significance: Players could achieve "common victories" by capturing certain numbers of pieces, or "proper victories" by arranging pieces in arithmetic, geometric, or harmonic proportions.

The three proper victories — small, big, and great — require arranging at least 3 of your pieces in a "coordinating progression" explained in detail later. For now, we simply list their names and spiritual meanings.

- Arithmetic progression (e.g., 3, 5, 7): representing justice and equity.
- Geometric progression (e.g., 2, 4, 8): symbolizing balance and proportion.
- Harmonic progression (e.g., 3, 5, 15): invoking celestial music and the soul's harmony. Some medieval authors called this a "musical progression" but harmonic progression is the most commonly used terminology.

All of this will be discussed in more detail later. For now, note that the Pythagoreans believed that achieving a "great victory" is not just a tactical feat but a spiritual one as well, akin to uniting body, soul, and spirit in perfect balance. These higher victory conditions represented spiritual enlightenment - the ability to perceive and create cosmic harmony.



1.4 In mathematics education

As mentioned, rithmomachia emerged from monastery and cathedral schools in the 11th century. One purpose was to teach Boethian number theory, a cornerstone of the medieval *quadrivium* (the four mathematical arts: arithmetic, geometry, music, and astronomy) which was the foundation of the educational system at the time.

Ironically, Boethius lived in the final years of the Western Roman Empire. After its fall, it took hundreds of years to rebuild the system of mathematical education that was in place. Monasteries and cathedral schools contributed to the development of the new educational system. Of course, after the fall the influence of Islamic scholarship in the Iberian Peninsula was significant as well. In particular al-Kwarizimi (780-850 C.E.), who introduced algebra and the Hindu-Arabic numeral system, had an influence lasting until today, while the quadrivium curriculum supported by Boethius faded away by the end of the middle ages. None-the-less, Boethius' text (the above-mentioned Latin translation of [Ni00]) was used for hundreds of years after his death – not just in these schools but many others in Europe. Indeed, one of the earliest universities to arise from the dust after the fall of the Western Roman Empire was the University of Paris, which coalesced, about 600 years after Boethius' lived, from an assimilation of the cathedral schools centered around Notre Dame.

Many historians believe the game was first described (and possibly invented by) a German monk named Asilo around 1030 C.E. some believe that this monk was none other than Adalbero, nominated by Henry III in 1045 to be the highly influential Bishop Adelbero of Würzburg [Mo01]. Since he was elevated to sainthood by Pope Leo XIII centuries later in 1883, he is now known as *Saint Adalbero*.

While Asilo is a fascinating character, he was aided in the development of rithmomachia by a contemporary monk, Herman the Lame (1013-1054), who may even be more fascinating. Also known as Hermannus Contractus and as Blessed Hermann of Reichenau, Herman the Lame was an 11th-century Benedictine monk and scholar. According to wikipedia, he was born with a cleft palate, cerebral palsy, and possibly had either amyotrophic lateral sclerosis or spinal muscular atrophy. Since the age of 20, he lived at the Abbey of Reichenau. In spite of his difficulties in movement and even in speaking, Herman became literate in several languages, and contributed scholarship to all four arts of the quadrivium. Herman is believed to have refined Asilo's original rules, writing one of the earliest treaties on the game [MacT]. Despite the fact that these two scholars lived only about 200 miles from each other, it's not known if they ever met in person. In fact, some historians suggest Herman the Lame invented rithmomachia before Asilo [SE11].

Who exactly invented the game may be forever lost to the sands of time, but it is known that medieval monastic teachers believe that by playing the game of rithmomachia the student engages with the mathematical principles that govern the universe.

Figure 3: Who invented rithmomachia ? Asilo as Bishop Adalbero of Würzburg (left), Blessed Hermannn of Reichenau (right)

Virtue quotation

Indeed, de Boissière emphasized how the ...

> ... game appeals to the weaker sides of human nature that seek diversion and entertainment; at the same time it leads the player to a more virtuous character.
>
> - Ann Moyer [Mo01], page 91.

Influenced by the works of Boethius (like *De institutione arithmetica*), medieval scholars took a Neoplatonic view of mathematics. To them, numbers weren't just abstract quantities, they were fundamental principles governing divine order, harmony, and the structure of reality. Understanding numerical relationships was seen as a way to understand the mind of God and the cosmos.

To help the student improve this numerical understanding, the teachers encourage students to play rithmomachia. Consider the rithmomachia rules for capture. The various capture methods (numbering, addition, subtraction, multiplication, division, siege) are direct exercises in calculation and recognizing numerical relationships (equality, sum, difference, product, ratio, spatial enclosure). They force the player to constantly analyze the numerical properties of the pieces and their spatial relationships.

Rithmomachia was a widely used instructional tool to teach arithmetic in monasteries, universities and cathedral-based grade schools in the later half of the middle ages. Unfortunately, it appears that the numerous versions of the game, with variations from one kingdom to the next, may have contributed to its slow acceptance, while chess was becoming more popular. Ultimately, the decline of rithmomachia was more a result of wide-spread change in the curriculum for mathematics education in universities. Indeed, around the 1600s, university curriculum we decided more by subject matter experts and less by authority figures (see chapter 5 in [Mo01]). Without the need to teach Boethian number theory and Pythagorean philosophy, there was no reason for teachers to promote the virtues of rithmomachia to their students.

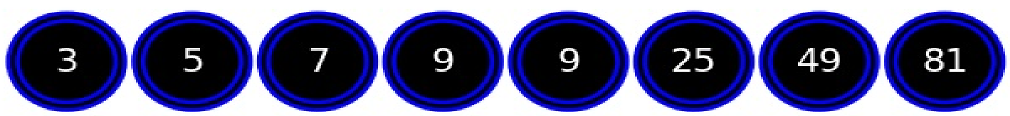


Figure 4: Rithmomachia Black circles pieces in a line

2 The battlefield

We shall adopt the *game board* as a 8×16 board. It may be checkered like a chess board or plain like a go board. Following military terminology first introduced in the 1500s, there are eight *files* and 16 *ranks*. On a rithmomachia board, the ranks are lettered (a-p) while the files are numbered (1-8). The ranks `a-h` comprise *Even/White territory*, while the ranks `i-p` comprise *Odd/Black territory*.

																1
																2
																3
																4
																5
																6
																7
																8
a	b	c	d	e	f	g	h	i	j	k	m	m	n	o	p	

Coordinate notation

An alternative to the algebraic notation is the coordinate-based notation shown in Figures 7 and 8. Imagine the rithmomachia board as being a grid in the 1st quadrant of the Cartesian xy plane ranging from 0 to 15 in the x-direction and 0 to 7 in the y-direction. We order the coordinates in this grid as though they were entries in an 8×16 matrix, with the count starting at 0 (instead of starting at 1).
Example: A White circle of value 6 located in the upper left-hand corner is denoted $C6{:}(0,0)$ or, in exponential notation, $C^6(0,0)$. Another option is to use the circle symbol font: $\bigcirc^6(0,0)$.

A schematic for how the armies are initially arranged on the battlefield before battle commences is given below.

Initial Setup

White/Even's Back Ranks:

Rank d	xxxx	xxxx	C008	C006	C004	C002	xxxx	xxxx
Rank c	T081	T072	C064	C036	C016	C004	T006	T009
Rank b	S153	P091	T049	T042	T020	T025	S045	S015
Rank a	S289	S169	xxxx	xxxx	xxxx	xxxx	S081	S025

(P denotes pyramid, T triangle, C circle, S square)

Black's setup mirrors this on the opposite side with its own set of values.

Black/Odd's Back Ranks:

Rank m	xxxx	xxxx	c009	c007	c005	c003	xxxx	xxxx
Rank n	t100	t090	c081	c049	c025	c009	t012	t016
Rank o	p190	s120	t064	t056	t030	t036	s066	s028
Rank p	s361	s225	xxxx	xxxx	xxxx	xxxx	s121	s049

(p denotes pyramid, t triangle, c circle, s square)

The symbols for the pieces will be described below.

3 The armies

Initially, both Black/Odd and White/Even are provided with a company of twenty-four pieces, each divided up into 8 platoon-like sets (to borrow the battlefield lingo):

- eight *circles* (denoted C or ○ for a white circle, c or ● for a black circle, when the color must be specified),
- eight *triangles* (denoted T or △ for white, t or ▲ for black), and
- eight *squares* (denoted S or □ for white, or s or ■ for black),
- For each side, exactly one of the square pieces is distinguished and called a *pyramid*, denoted $P = \diamond$ (or $p = \blacklozenge$). This type of piece is more powerful, and more complicated, than the others but will be explained in complete detail later.

The position of the pieces during a particular moment in a game will be called the (current) *state* or the *game state*.

In chess, the files are named after the back-rank pieces initially occupying them: the queen rook file, the queen knight file, and so on. Likewise, in rithmomachia, we can name the file after the *most valuable* piece initially occupying it:

`289`, `169`, `49`, `42`, **56**, **64**, **225**, **361**.

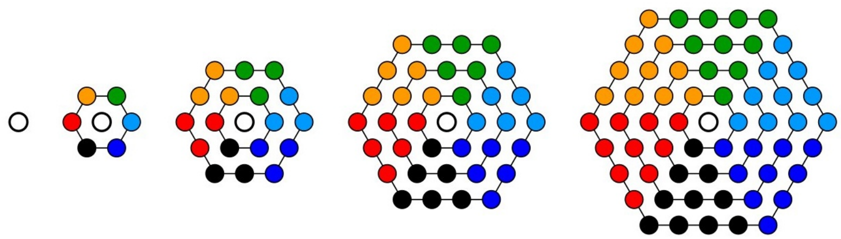
Note: the number is in **bold** font if it is a Black piece, `Courier` font if it is a White piece. Half of these are White/Even and half are Black/Odd!

Some other fun facts about the file numbers:

1. 289 is
   - a perfect square ($= 17^2$),
   - a sum of perfect cubes ($1^3 + 2^3 + 4^3 + 6^3$), and
   - is the sum of the first 5 numbers raised to themselves ($0^0 + 1^1 + 2^2 + 3^3 + 4^4$).

2. 169 is

   - a perfect square ($= 13^2$),
   - a centered hexagonal number.



   A centered hexagonal number H_n counts the number of equally spaced dots inside a hexagon. There is a formula for them: $H_n = n^3 - (n-1)^3$.

3. 49 is

   - a perfect square ($= 7^2$),
   - the smallest triple of three squares in arithmetic succession $(1, 25, 49)$.

4. 42 is

   - not a perfect square, but "near" one: it's the product of consecutive integers, $6 \cdot 7$, one of which is perfect!
   - Anyone who has heard of Douglas Adams' book *The Hitchhiker's Guide to the Galaxy* knows all that is needed to know about the number 42!

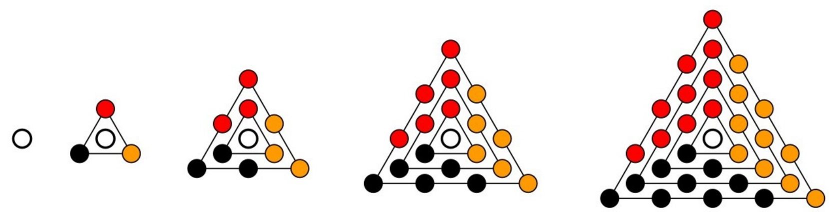
5. 56 is

   - the sum of the first six triangular numbers (so counts the equally spaced dots in a regular tetrahedron whose base has length 6),
   - twice a perfect number ($= 2 \cdot 28$).

   Plutarch [Pl00] reported the Pythagoreans associated a regular polygon of 56 sides with Typhon (one of the deadliest creatures in Greek mythology).

6. 64 is

    - a perfect square,
    - a perfect cube,
    - a centered triangular number.



The centered triangular numbers are constructed by placing a dot in the center and all its other dots surrounding the center in successive equilateral triangular layers.

> **centered triangular number**
>
> The triangular numbers are constructed inductively in a different way: by adding a slightly larger base (depicted as dots along a line segment).
> If C_n is the nth centered triangular number and T_n is the nth triangular number then there is a simple formula relating them: $C_n = 3T_n + 1$.

7. Note 225 is (i) a perfect square ($= 15^2$), (ii) the square of a triangular number.

8. Note 361 is (i) a perfect square ($= 19^2$), (ii) a perfect cube, (iii) a centered triangular number.

4 The rules

This history and philosophical underpinnings of the game presented in the previous section makes it clear that there were several design decisions made by the inventor (or inventors) or rithmomachia that are very spiritual in nature. Do today's players believe that by playing rithmomachia they become closer to God? Maybe they don't, but it seems the game was designed to be entertaining to those who did.

Of all the many versions of rithmomachia, this section only records the rules and setup for the version described by Claude de Boissière in 1556. Not a lot is known about Claude de Boissière, also known as Claudius Buxerius. Born around 1500 near Grenoble, he was an influential French scholar active in promoting mathematical education. He wrote and published texts on various subjects, including poetry, music, and astronomy. His most renown work is [dB56], translated by Richards in [Ri46].

The current section delves into the fundamental rules governing the game of rithmomachia, based on the version described by de Boissière. Sections §§4.1-4.2 detail the game's physical components, covering the specifics of the board and the various pieces involved. Following this, §4.3 explains the initial arrangement of these pieces on the board. The subsequent sub-sections will describe the distinct ways in which each piece can move (§4.4) then the various methods for capturing an opponent's pieces (§4.5). Finally, §4.6 describes the conditions that determine victory in a rithmomachia game.

4.1 Initial setup

Like chess or checkers, rithmomachia is a board game played between two people, called Black (or Odd) and White (or Even). These different names (White and Even, for example) will be used interchangably. The two players alternate turns, each turn consisting of a move of a friendly piece and possibly captures of enemy pieces. According to de Boissière, White/Even has the first turn. Other versions of rithmomachia state Black/Odd moves first. Here, we give an overview of the game play process.

To describe these rules more clearly, we distinguish between a move and a capture (where no friendly piece moves but an enemy piece is taken off the board). Details of how the pieces move will be described in §4.4 and how they capture will be described in §4.5.

What is a player's turn?

A *turn* consists of three stages.

- Inital captures, if any.

  For instance, such captures could be the result of the opponent's previous move. In any case, more than one capture is allowed.

- A legal move.

  Exactly one move is made per turn, onto an unoccupied position on the board. This space could be the one occupied by an enemy piece just captured.

- Final captures, if any.

  This could be a capture that was made possible by move just made.

If it is a player's turn and no legal move can be made, then that player *loses* the game. This is in contrast to chess, where a stalemate results in a *draw*.

There are no draws in this version of rithmomachia.

4.2 Board and pieces

Each piece has a value. These values are tabulated in Figures 6 and 5.

Both sides have piece values 9, 16, 25, 36, 49, 64, 81 in common. These values, all square numbers, are visible in the diagram of the initial board setup in Figure 7 and in Figure 8.

These are the pieces and their values. Next, we learn about some of the numerical relationships among a piece and its neighbors in their initial placement on the board.

4.3 Setting up the pieces

The chess-like notation for the rows and columns used in Figure 7 is borrowed from the book Codenotti-Resta [CR22]. Unlike chess, there is no standard-

piece					formula
circles	9	7	5	3	x
circles ●	81	49	25	9	x^2
triangles	90	56	30	12	$x(x+1)$
triangles ▲	100	64	36	16	$(x+1)^2$
squares	**190**◆	120	66	28	$(x+1)(2x+1)$
squares ■	361	225	121	49	$(2x+1)^2$

Figure 5: Black pieces and their values, which total 1752. The black pyramid is the square 190. You can imagine stretching this table over the Black side of the board to see how the pieces are initially placed. Note all the values of the back rank of the Black/Odd pieces are odd numbers, while all the values in the second-to-the-last rank of pieces are *even* integers.

piece					formula
circles	8	6	4	2	y
circles ○	64	36	16	4	y^2
triangles	72	42	20	6	$y(y+1)$
triangles △	81	49	25	9	$(x+1)^2$
squares	153	15	**91**◇	45	$(y+1)(2y+1)$
squares □	289	169	81	25	$(2y+1)^2$

Figure 6: White pieces and their values, which total 1312. The White pyramid is the square 91. Note all the values of the back rank of the Even/White pieces are *odd* integers.

ized notation yet for rithmomachia but this Codenotti-Resta notation is a place to good start.

It's natural to wonder how these piece values were selected. By design,

p	■$_1$49	■121					■225	■361
o	■28	■66	▲36	▲30	▲56	▲64	■120	◆190
n	▲16	▲12	●$_2$9	●25	●49	●81	▲90	▲100
m			●3	●5	●7	●$_1$9		
l								
k								
j								
i								
h								
g								
f								
e								
d			○8	○6	○$_1$4	○2		
c	△81	△72	○64	○36	○16	○$_2$4	△6	△9
b	□153	◇91	△49	△42	△20	△25	□45	□15
a	□289	□169					□81	□$_1$25
	1	2	3	4	5	6	7	8

Figure 7: The initial state of rithmomachia, with ◇(at $b2$, in algebraic notation) and ◆(at $o8$) as the pyramids. Note there are duplicate pieces (eg, two White circles of value 4 denoted ○$_1$4 and ○$_2$4), distinguished using subscripts when necessarly. All these pieces, including the duplicates, will be listed in §4.3.1 below.

the game contains some interesting patterns in the initial setup in Figure 8. The arithmetical patterns in the setup of the pieces is displayed in the next two tables, presented next.

Start with the top row of pieces in the initial position – namely, $8, 6, 4, 2$ for White and $9, 7, 5, 3$ for Black. For simplicity, we'll denote this row by

a, b, c, d. The following lemma basically says very other piece value is easily determined from a, b, c, d.

Lemma 1. Each initial piece value can be computed using a polynomial in the values of the forward-most piece values.

Proof. This follows from Figures 5 and 6, but will also be established in the tables below. $\square$

To get the value of the pieces in the second row, compute these (mostly) squares:

		a	b	c	d		
$(a+1)^2$	$a+a^2$	a^2	b^2	c^2	d^2	$d+d^2$	$(d+1)^2$
$\dots$	$\dots$					$\dots$	$\dots$

Table: top two rows

Rename these values of the 2nd row as $e, f, \dots, \ell$ for simplicity. The numerical relationships in the third row are also easy to describe:

		a	b	c	d		
e	f	g	h	i	j	k	ℓ
m	n	o	p	q	r	s	t
u	v	0	0	0	0	w	x

Table: all four rows

We know already,

$$e = (a+1)^2, \quad f = a^2 + a, \quad g = a^2, \quad h = b^2,$$
$$i = c^2, \quad j = d^2, \quad k = d^2 + d, \quad \ell = (d+1)^2,$$

but the remaining expressions are easy to compute as well:

$$m = (a+1)(2a+1), \quad n = (b+1)(2b+1), \quad o = (b+1)^2, \quad p = b^2 + b,$$

$$q = c^2 + c, \quad r = (c+1)^2, \quad s = (c+1)(2c+1), \quad t = (d+1)(2d+1).$$

We leave the values of the pieces at the bottom to the reader.

Question 1. *Can you find the arithmetical relationships that determine the numbers u, v, w, x in the bottom row of the above table? Here, for example, $u = 289, \dots, x = 25$ if you are White and $u = 361, \dots, x = 49$ if you are Black. What algebraic formula determines them from the other entries? (Hint: they are all perfect squares! The answer is at the end, just before the references.)*

The symmetry and arithmetical properties of the setup of the rithmomachia pieces is remarkably beautiful, isn't it? The interested reader can find another discussion of the numerical relationships between the piece values on pages 74-75 of [SE11].

Algebraic notation

Each of the 128 *coordinate positions* on the board can be described by a pair, which we juxtapose for simplicity: $a1$, $a2$, ..., $p8$. The shape of the pieces can be described by its first letter: C, P, S, T for the Even/White pieces and c, p, s, t for the Odd/Black ones. The value of a piece can be described by a list of positive integers[a].
Example: A Black circle of value 3 at location $h5$ is denoted $c3h5$, but we sometimes use exponential notation, c^3h5. Another option is to use the filled circle symbol font: $\bullet^3 h5$.

[a]Except the case of the multi-valued pyramid piece and its subpieces, which is described in detail in §§4.4.3, 4.5.7 below, all these lists have only one element, na,ely the value of the piece itself.

The pieces are all listed in §4.3.1 below, including all duplicates and component subpieces of the pyramid pieces.

4.3.1 Enumerating the pieces

If we count the White pieces, including the duplicates, the pyramid, and all its sub-pieces, we get 30:

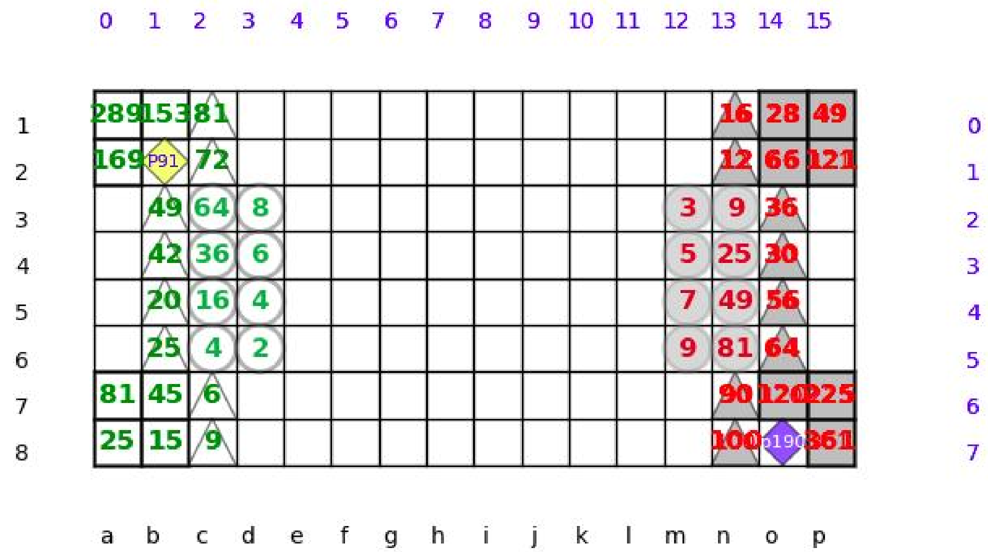


Figure 8: Rithmomachia initial position, where 91 (at $(1,1)$ in coordinate notation) is the value of the white pyramid and 190 (at $(7,14)$ in coordinate notation) is that of the black pyramid. This diagram matches that of de Boissière's, as shown in [Ri46], Figure 8.

$$S^{289},\ S^{169},\ S^{153},\ P^{91},\ S^{81},\ T^{81},\ T^{72},\ C^{64},\ T^{49},\ S^{45},\ T^{42},$$
$$C^{36},\ S^{36},\ S_1^{25},\ S_2^{25}, T^{25},\ \ T^{20},\ C^{16},\ S^{16},\ S^{15},\ T^9,\ S^9,\ C^8,$$
$$T^6,\ C^6,\ C_1^4,\ C_2^4,\ S^4,\ C^2,\ S.$$

(We use subscripts to notationally distinguish the two White circles of value 4, and the two White square of value 25.) On the other hand, the total number of distinct values used by the Even/White pieces is, due to the duplicates, only 20.

Duplicate values for the Even/White pieces

White has three 4s, two 6s, two 9s, two 16s, two 25s, two 36s, and two 81s. Other piece values are unique to the color. Here is a histogram plot:

	○						
	○	○	○	○	○	○	○
	○	○	○	○	○	○	○
piece value	4	6	9	16	25	36	81

A way to avoid using subscripts in the notation, yet still eliminate ambiguities caused by the repetition of piece values, is to include the initial coordinate of the piece:

- There are 14 □'s (counting duplicates and the pyramid as a square):

$$S^{289}a1,\quad S^{169}a2,\quad S^{153}b1,\quad P^{91}b2,\quad S^{81}a7,\quad S^{45}b7,$$
$$S^{36}b2,\quad S^{25}a8,\quad S^{25}b2,\quad S^{16}b2,\quad S^{15}b8,\quad S^{9}b2,\quad S^{4}b2,\quad S^{1}b2.$$

- There are 8 △s:

$$T^{81}c1,\quad T^{72}c2,\quad T^{49}b3,\quad T^{42}b4,\quad T^{25}b6,\quad T^{20}b5,\quad T^{9}c8,\quad T^{6}c7.$$

- There are 8 of the ○:

$$C^{64}c3,\quad C^{36}c4,\quad C^{16}c5,\quad C^{8}d3,\quad C^{6}d4,\quad C^{4}c6,\quad C^{4}d5,\quad C^{2}d6.$$

If we count the Black pieces, including the duplicates, the pyramid, and all its sub-pieces, we get 29:

- There are 14 □'s

$$\blacksquare 361,\ \blacksquare 225,\ \blacksquare 190,\ \blacksquare 121,\ \blacksquare 120,\ \blacksquare 66,\ \blacksquare 64,$$
$$\blacksquare_1\, 49,\ \blacksquare_2\, 49,\ \blacksquare 36,\ \blacksquare 28,\ \blacksquare 25,\ \blacksquare 16,$$

Note there are 2 black squares with value 49.

- There are 8 of the ▲'s:

  ▲100, ▲90, ▲64, ▲56, ▲36, ▲30, ▲16, ▲12,

- There are 8 of the ●'s:

  ●81, ●49, ●25, ●$_1$ 9, ●$_2$ 9, ●7, ●5, ●3.

  Note there are 2 black circles with value 9.

On the other hand, the total number of distinct values used by the Black/Odd pieces is, due to the duplicates, only 22.

Duplicate values for the Odd/Black pieces

Black has two 9s, two 16s, two 25s, two 36s, three 49s, and two 64s. Other piece values are unique to the color. Here's a histogram plot:

					○	
	○	○	○	○	○	○
	○	○	○	○	○	○
piece value	9	16	25	36	49	64

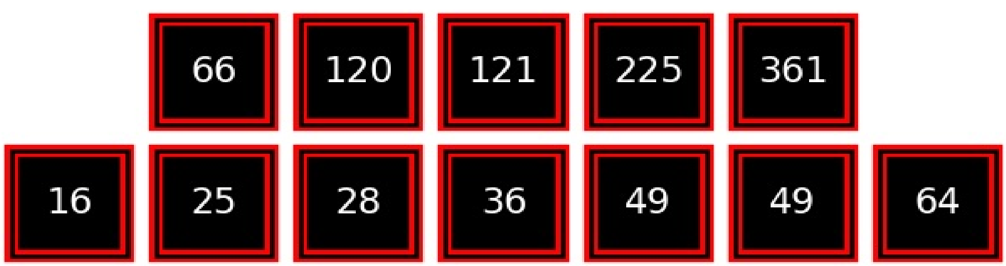


Figure 9: Rithmomachia Black square pieces

4.4 How the pieces move

The rules for how pieces move are pretty simple.

4.4.1 Movement notation

During a player's turn, only one move (as described in this section) can be made, but multiple captures are possible, and these can be performed before and/or after the move. Captures are described in a later section.

Move notation

Based on notation used for chess, one simple coordinate-based notation for a move would be something like this:

```
move = (shape+value):(starting coord.)-(ending coord.).
```

For example, in coordinate notation $C2{:}(5,3)-(6,3)$ describes moving the white circle piece with value 2 from the position/coordinate $(5,3)$ to $(6,3)$. In the more compact algebraic notation, this move would be $C2d6-d7$, or C^2d6d7, or even $\bigcirc^2 d6d7$.

For another example of a move in coordinate notation, see Figure 10 for the move $S^{81}{:}(6,0)-(3,0)$. That same move in the more chess-like notation of Figure 7, would be written as $S^{81}a7-a4$ or $\square^{81}a7a4$. See the caption in Figure 8 for more coordinate examples.

4.4.2 Movement rules

While it may seem complicated at first sight, the rules are pretty simple and easy to understand.

- The players move alternately, each move must land on an empty space on the game board.

  Other versions of rithmomachia list this rule: "No move can return the game to a previous state." This is called the *no repetition rule*. This one is not used by de Boissiére, so it is ignored here.

- No piece is allowed to jump over another, a move possible in some other versions of rithmomachia. Howeever, de Boissière did not approve, calling the chess knight which can jump over other pieces, "the mad warrior" (see page 199 in [Ri46]).

- Each shape has a different movement (although, the pyramid piece is skipped over for the time being).
- The movement of a piece doesn't depend on its value.
- A circle piece moves exactly one space vertically or horizontally (but not diagonally) in any direction.

		○		
	○	*	○	
		○		

Table: Possible ○, ● moves from *

- A triangular piece moves two spaces in any horizontal or vertical direction.

		△		
△		*		△
		△		

Table: Possible △, ▲ moves from *

- A square piece moves exactly three squares but only vertically or horizontally.

			■			
■			*			■
			■			

Table: Possible □, ■ moves from *

- For clarity we repeat: in this version of Rithmomachia no piece can move (or capture) diagonally.

Remark 1. A remark on a parity condition that these rules determine:

- As a consequence of these conditions, a triangle piece of either color with initial position in column 1 can never move to an even column (since it must move 2 spaces, hence this triangle can only move to an odd column).
- Likewise, if it's initially in column 2 it can never move to an odd column.

Question 2. *A similar condition to that in Remark 1 holds for the squares. Can you find it?*

When a piece cannot move (due to the positions of other pieces) then it is *blocked*.

4.4.3 The pyramid

In the de Boissière's set up, the pyramid piece moves just like the square piece. In all versions of rithmomachia, the pyramid piece can move to an unoccupied coordinate/position if and only if one of its component sub-pieces can legally move to that position. Since some versions allow circle and triangular subpieces of the pyramid (e.g., [Ne98] and [DG21]), such versions endow more flexibility of movement on the pyramid than we do here.

The *white pyramid*, is initially set to be S^{91}, the white square with (total) value 91. Create the following six white *sub-pieces*, depicted in Figure 11.

■49	■121					■225	■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81	▲90	▲100	n
		●3	●5	●7	●9			m
								l
								k
								j
								i
								h
								g
								f
								e
		○8	○6	○4	○2			d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169		□81				□25	a
1	2	3	4	5	6	7	8	

Figure 10: Rithmomachia position after the move □^{81}a7a4.

□36, □25, □16, □9, □4, □1,

which are called the *sub-piece components* of the pyramid. These are not separate pieces, but can be involved with captures, as we will explain in more detail in §4.5.7.

Imagine a layer of 36 equally spaced dots arranged in a 6×6 grid. Centered over that, imagine a layer of 25 equally spaced dots arranged in a 5×5 grid. Continuing in this fashion, on to of that stack a 4×4 layer, a 3×3 layer, a 2×2 layer, and finally a dot on top. This is one way to imagine the White

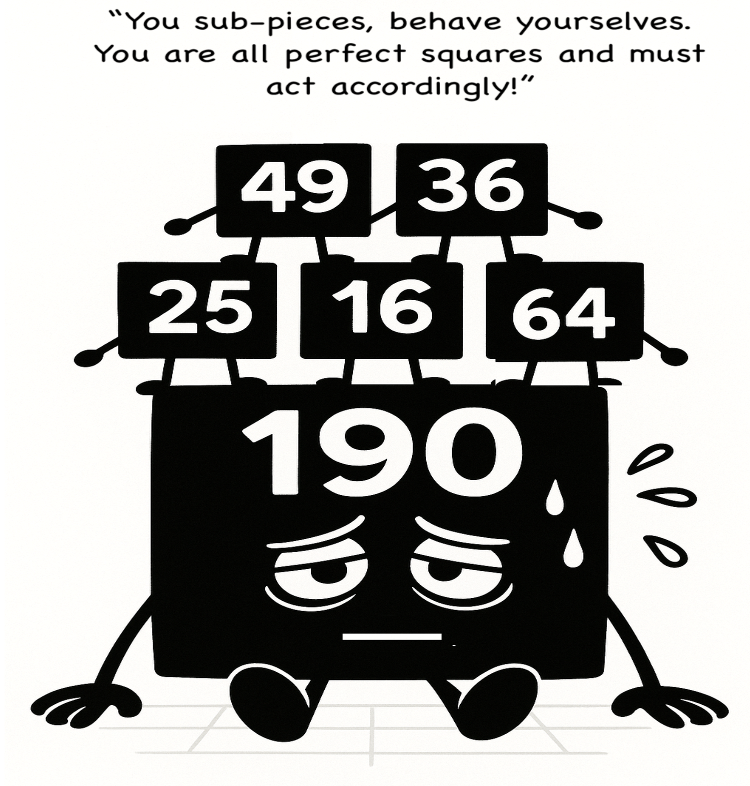


Figure 11: **rithmomachia** pyramids are multi-valued because they have sub-pieces. Thanks chatGPT for the visual humor!

pyramid.

The *black pyramid*, is initially set to be s^{190}, the black square 190. Create the following five black *sub-pieces*:

$$\blacksquare 64, \blacksquare 49, \blacksquare 36, \blacksquare 25, \blacksquare 16,$$

which are called the *components* of the pyramid. In the polynomial ring with variables c, C, t, T, s, S,

$$R = \mathbb{Z}[c, C, t, T, s, S],$$

A sub-piece component only can be used (and potentially captured by White) as part of the pyramid. The list of values of p is, by design, the sequence of five consecutiive squares whose total is 190 (namely, $16, 25, 36, 49, 64$).

Imagine a layer of 64 equally spaced dots arranged in an 8×8 grid. Centered over that, imagine a layer of 49 equally spaced dots arranged in a 7×7 grid. Continuing in this fashion, on to of that stack a 6×6 layer, a 5×5 layer, and finally a 4×4 layer. This is one way to imagine the Black pyramid.

Pyramid decompositions

Algebraically, White's pyramid can be initially represented as

$$P = S^{36} + S^{25} + S^{16} + S^{9} + S^{4} + S^{1},$$

or

$$\Diamond = \square^{36} + \square^{25} + \square^{16} + \square^{9} + \square^{4} + \square^{1},$$

and its (list of) values as $[1, 4, 9, 16, 25, 36]$ and its total 91.
Black's pyramid can be initially represented as a polynomial belonging to R,

$$p = s^{64} + s^{49} + s^{36} + s^{25} + s^{16},$$

or

$$\blacklozenge = \blacksquare^{64} + \blacksquare^{49} + \blacksquare^{36} + \blacksquare^{25} + \blacksquare^{16},$$

and its (list of values) as $[16, 25, 36, 49, 64]$ and its total 190.

The Pythagoreans believed that elucidating such harmonious arithmetical relationships is spiritually significant.

The rules on capturing a pyramid, or one of its component sub-pieces, will be explained in §4.5.7.

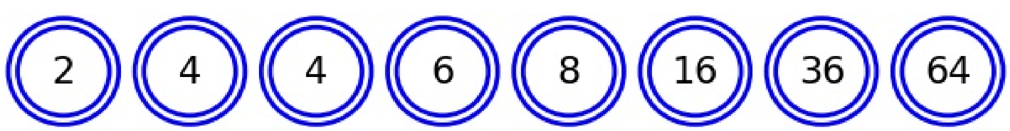


Figure 12: Rithmomachia White circles pieces in a line

4.5 How the pieces capture

As mention, a player's turn consists of captures and a piece move. How do captures work?

We follow de Boissiére's set-up, as discussed in [Ri46].

How Pieces Capture

Capture of an enemy piece does not mandate a friendly piece move but requires permanent removal of the captured piece or subpiece. In particular, a captured piece can never be placed back on the board, even as the opposite color. If a pyramid sub-piece is captured, then that subpiece is removed and the value of the pyramid is deducted accordingly. Captures are not mandatory.

A natural question arises: What about repeated captures? They are allowed in checkers but not in chess. What about for rithmomachia?

Question 3. *Assume you are playing a game and the following scenerio (or game state to use the technical term) arises:*

(a) you capture an opponent piece, and

(b) its removal from the board exposes a second opponent piece for capture (by the same friendly piece or by some other).

Can you take it?

There is no rithmomachia rule that stops you from capturing that second opponent piece as well. If you are lucky enough to land in such a game state, the captures are allowed. When at least two pieces are captured in this way, it is called a *chain capture.*

Like chess, captures in rithmomachia are not mandatory. There are several types:

- capture by *numbering* (or *encounter*),
- by *addition* (or *ambush*),
- *subtraction* (also called an *ambush*),
- *multiplication* (or *assault*),

- *division* (also called an *assault*),
- and *siege* (or *surrounding*).

We will go into more detail about each of these below.

If we ignore the pyramid pieces for the moment, a captured piece is *removed* from the board (and cannot be returned or replaced or swapped). To be perfectly clear: the capturing player need not move their capturing piece (e.g., to replace the captured piece, as in chess). Captures involving the pyramids are treated separately below, but to give a hint: if a subpiece of a pyramid is captured then that subpiece is removed but the pyramid is not removed.

The captures by numbering, by addition, and by subtraction all have the special property called *local captures*. That is, such a capture is performed "up close" in the sense that the capturing piece(s) can, in one move, land on captured piece.

Main "local" capture types

- **Numbering:** A piece captures an enemy piece if it can land on the enemy's square and both pieces have the **same value**.
- **Addition/Subtraction:** Two friendly pieces capture an enemy piece if the **sum/difference** of their values equals the enemy's value, and each friendly piece can legally land on the enemy's square.

On the other hand, capturing by multiplication or by division has a different property, called *non-local*: such a capture can be performed at a distance not equal to the move distance.

Main “non-local” capture types

- **Multiplication:** A friendly piece (value v) captures an enemy piece (value $m \cdot v$) if there are exactly m vacant squares in a straight orthogonal line between them ($m > 0$).
- **Division:** A friendly piece (value $d \cdot v$) captures an enemy piece (value v) if there are exactly d vacant squares in a straight orthogonal line between them ($d > 0$).
- **Siege:** An enemy piece is captured if it's surrounded by opposing pieces (or board edges) such that it has no legal moves.

An example of what this means is this: Suppose, during White's turn, one or more of Black's pieces is captured. White removes the capture(s) and now *must* make a move of some piece to an unoccupied board position (see [Ri46], pp 202-203). This is in contrast with chess - in chess, a capturing piece *must* replace the captured piece.

capture type	description
numbering	1 friendly takes 1 enemy, local
addition	2 friendlies take 1 enemy, local
subtraction	2 friendlies take 1 enemy, local
multiplication	1 friendly takes 1 enemy, non-local
division	1 friendly takes 1 enemy, non-local
siege	many friendlies takes 1 enemy, local

Table: piece capture types

4.5.1 Numbering

If a piece can move to a position where it lands on an enemy piece with the same number, then the enemy piece may be captured.

For a numbering capture to occur, the values of two opposing pieces must be equal. As a matter of fact, here are the only values for which there is a piece of each color with that value: 9, 16, 25, 36, 49, 64, 81.

Capture by numbering notation

A capture by numbering could be written something like this:

`(attacking piece+value:coord.)=(captured piece+value:coord.).`

However, often the context is clear and we simply write

`(attacking piece+value)x(captured piece+value).`

Lemma 2. *Given all pieces are on the board, there are* 21 *possible White captures Black by numbering.*

Proof. The simplest verification is to simply list all such captures.

First we list White piece/Black piece pairs that satisfy in theory capturing by numbering. Then we take parity conditions into account. The initial coordinates of the pieces are included below to avoid any ambiguities.

$$S^{81}(6,0)=c^{81}(5,13),\quad T^{81}(0,2)=c^{81}(5,13),\quad C^{64}(2,2)=s^{64}(7,14),$$
$$C^{64}(2,2)=t^{64}(5,14),\quad T^{49}(2,1)=s^{49}(0,15),\quad T^{49}(2,1)=c^{49}(4,13),$$
$$T^{49}(2,1)=s^{49}(7,14),\quad C^{36}(3,2)=s^{36}(7,14),\quad C^{36}(3,2)=t^{36}(2,14),$$
$$S^{36}(1,1)=s^{36}(7,14),\quad S^{36}(1,1)=t^{36}(2,14),\quad S^{25}(1,1)=s^{25}(7,14),$$
$$S^{25}(1,1)=c^{25}(3,13),\quad S^{25}(7,0)=s^{25}(7,14),\quad S^{25}(7,0)=c^{25}(3,13),$$
$$T^{25}(5,1)=s^{25}(7,14),\quad T^{25}(5,1)=c^{25}(3,13),\quad C^{16}(4,2)=s^{16}(7,14),$$
$$C^{16}(4,2)=t^{16}(0,13),\quad S^{16}(1,1)=s^{16}(7,14),\quad S^{16}(1,1)=t^{16}(0,13),$$
$$T^{9}(7,2)=c^{9}(2,13),\quad T^{9}(7,2)=c^{9}(5,12),$$
$$S^{9}(1,1)=c^{9}(2,13),\quad S^{9}(1,1)=c^{9}(5,12).$$

The parity conditions mentioned in Remark 1 and Question 2 tell us that the following are impossible,

$$S^{25}(7,0)=s^{25}(7,14),\quad S^{16}(1,1)=s^{16}(7,14),$$
$$S^{36}(1,1)=s^{36}(7,14),\quad S^{25}(1,1)=s^{25}(7,14).$$

but don't rule out the remaining ones. This leaves 21. □

Question 4. *Suppose it's White turn and a White triangle is able to capture an Black square by numbering. By the movement rules, this implies there is exactly one empty coordinate position between these two opposing pieces.*

Does this also means that, if it were instead Black's turn to move, the enemy square can capture the triangle by numbering? (Hint, hint[1] :-)

None-the-less, it turns out there are also 21 Black captures White by numbering.

Before continuing, we need more notation.

Modulus notation $\equiv$

The modulus notation, introduced by Carl Frederick Gauss in the 1800s, looks a little strange at first sight but it's easy to work with. For integers $a, b, d > 0$, the expression

$$a \equiv b \pmod{d}$$

is pronounced "a is congruent to b modulo d" and simply means that d divides $b - a$. It's treated in practice more as an equality rather than a divisibility condition. For example, $13 \equiv 1 \pmod{12}$ simply says that "13 is congruent to 1 modulo 12". In non-technical language, that's the same as saying 13 o'clock is the same as 1 o'clock. Think of $a \equiv b \pmod{d}$ as "=" on the (analog) clock face with d numbers spaced equally clockwise around the face.

Remark 2. A remark on symmetry and parity considerations expanding on Remark 1.

Since a triangle moves by 2, if one starts at a coordinate position (i, j) then the only other positions it can possibly land on during the game are $(i + 2a, j + 2b)$, for some integers a and b. To get a visual of this, imagine the Rithmomachia board is checkered, like a chess board. The above identity implies that if a triangle started on a black square, it can never land on a white square.

Therefore, we have the following *Triangle Parity Lemma*.

[1] Indeed, the movement rules require *two* empty coordinate positions between them.

Lemma 3. *If we have one triangle at* (i_1, j_1) *and another triangle at* (i_2, j_2) *then one can land on the other if and only if*

$$i_1 \equiv i_2 \pmod 2 \quad \text{and} \quad j_1 \equiv j_2 \pmod 2.$$

Remark 3. Similarly, if we have one square at (i_1, j_1) and another square at (i_2, j_2) then one can land on the other if and only if

$$i_1 \equiv i_2 \pmod 3 \quad \text{and} \quad j_1 \equiv j_2 \pmod 3.$$

Therefore, we have the following *Square Parity Lemma.*

Lemma 4. *If we have one square at* (i_1, j_1) *and another square at* (i_2, j_2) *then one can land on the other if and only if*

$$i_1 \equiv i_2 \pmod 3 \quad \text{and} \quad j_1 \equiv j_2 \pmod 3.$$

In particular, the sub-piece S^{36} of the white pyramid can never capture the sub-piece s^{36} of the black pyramid since they have different coordinate parities.

4.5.2 Addition

In this capture, two pieces on the same side must satisfy these conditions:

- there is an enemy piece that either capturing piece can legally land on,
- the value of the enemy piece is the sum of the values of the two capturing pieces.

Then the enemy piece can be captured.

In particular, if White moves any piece of value x to a coordinate that is

(a) orthogonally next to a black circle piece with value y,

(b) orthogonally next to another black circle piece with value z,

(c) $x = y + z$,

then Black can capture this White piece on the next move.

	● ^{x}		
	□ $^{x+y}$		▲ ^{y}

Table: Addition: Black/Odd pieces of value x, y capturing a White piece of value $x + y$

Capturing by addition notation

A capture by addition could be written something like this:

`[(attacking piece1+value1:coord.)+(attacking piece2+value2:coord.))=(captured piece3+value3:coord.)`.

However, often the context is clear and we simply write

`[(attacking piece1+value1)+(attacking piece2+value2)]x(captured piece3+value3)`.

Example: In the table above, we have $[c^x(i-1,j)+t^y(i+2,j)]\times S^{x+y}(i,j)$, where i is the rank (row number) of the captured piece and j is the file (column number).

Another example of Black capturing by addition is depicted in Figure 13. It is written in coordinate notation as,

$$[c^3\texttt{(1,3)} + c^5\texttt{(2,4)}]\times C^8\texttt{(2,3)} \text{ (addition)}$$

and in algebraic notation, as

$$[c^3\texttt{d7} + c^5\texttt{e6}]\times C^8\texttt{d6} \text{ (addition)} .$$

There are at most 118 ways for White to capture Black by addition. To be perfectly clear, the list of the value combinations that allow for a capture by addition has length 118. As in the proof of Lemma 2, one must check whether parity conditions rule any of these captures out.

Some examples are listed below (using subscripts when there is ambiguity):

$$[S^{289}+T^{72}]\times s^{361},\quad [S^{153}+T^{72}]\times s^{225},\quad [P^{91}+T^{9}]\times t^{100},$$
$$[P^{91}+S^{9}]\times t^{100},\quad [S^{81}+T^{9}]\times t^{90},\quad [S^{81}+S^{9}]\times t^{90},$$
$$[T^{81}+T^{9}]\times t^{90},\quad [T^{81}+S^{9}]\times t^{90},\quad \ldots .$$

There are at most 82 ways for Black to capture White by addition. Some examples are listed below:

$$[s^{225}+s^{64}]\times S^{289},\quad [s^{225}+t^{64}]\times S^{289},\quad [s^{120}+c^{49}]\times S^{169},$$
$$[s^{120}+s_1^{49}]\times S^{169},\quad [s^{120}+s_2^{49}]\times S^{169},\quad [s^{66}+s^{25}]\times P^{91},\quad \ldots .$$

Question 5. *What is the exact number of ways for Black to capture White by addition? Same question for White to capture Black by addition.*

4.5.3 Subtraction

Very similar to addition: two pieces on the same side must satisfy these conditions:

- there is an enemy piece that either capturing piece can legally land on,
- the value of the enemy piece is the difference of the values of the two capturing pieces.

Then the enemy piece can be captured.

Capturing by subtraction notation

A capture by subtraction could be written something like this:

```
[(attacking piece1+value1:coord.)-(attacking
piece2+value2:coord.))=(captured piece3+value3:coord.).
```

However, often the context is clear and we simply write

```
[(attacking piece1+value1)-(attacking
piece2+value2)]x(captured piece3+value3).
```

Example: In Figure 13, Black captures the piece $\bigcirc^2$ by subtraction,

■49	■121					■225	■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81	▲90	▲100	n
								m
								l
								k
								j
								i
								h
								g
								f
		●5			●9			e
	●3	○8	○6	○4	○2	●7		d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Figure 13: Rithmomachia position before a capture by addition and a capture by subtraction.

[c^9e3 - c^7e5]$\times C^2$d4 (subtraction).

or $(c^9 - c^7) \times C^2$.

There are at most 141 ways for White to capture Black by subtraction. Some examples are listed below:

$$[S^{289} - S^{169}] \times s^{120}, \quad [S^{289} - C^{64}] \times s^{225}, \quad [S^{169} - S^{153}] \times s^{16}(7, 14),$$
$$[S^{169} - S^{153}] \times t^{16}, \quad [S^{169} - T^{49}] \times s^{120}, \quad [S^{153} - T^{72}] \times c^{81},$$

$$[P^{91} - T^{42}] \times s_1^{49}, \quad [P^{91} - T^{42}] \times s_2^{49}, \quad [P^{91} - T^{42}] \times c^{49}, \quad \ldots$$

There are at most 119 ways for Black to capture White by subtraction. Some examples are listed below:

$$[s^{225} - t^{56}] \times S^{169}, \quad [s^{121} - s^{120}] \times S^{1}, \quad [s^{121} - c^{49}] \times T^{72},$$
$$[s^{121} - s_1^{49}] \times T^{72}, \quad [s^{121} - s_2^{49}] \times T^{72}, \quad [s^{121} - t^{30}] \times P^{91}, \quad \ldots$$

Question 6. *What is the exact number of ways for Black to capture White by subtraction? Same question for White to capture Black by subtraction.*

4.5.4 Multiplication

A friendly piece of value v can capture an enemy piece of value $m{\cdot}v$ *if* the number of vacant coordinates along an unoccupied straight line (vertical, horizontal) drawn between these two pieces is exactly $m > 0$. In the special case when $m = 1$ then capturing by multiplication coincides with capture by *numbering*, in which case it *is* necessary for the friendly capturing piece to be able to land on the enemy captured piece. However, outside of this special case, it is not generally necessary for the friendly capture-by-multiplication piece to be able to land on the enemy captured piece on their next move.

▲ x	$\leftarrow \ldots$	m	$\ldots \rightarrow$	□ $m \cdot x$

Table: Multiplication: Black piece of value x capturing from a distance a White piece of value mx

Capturing by multiplication notation

A capture by multiplication could be written something like this:

`[(attacking piece1+value1:coord.)*distance))`$\times$`(captured piece2+value2:coord.)`.

However, often the context is clear and we simply write

`[(attacking piece1+value1)*distance]x(captured piece2+value2)`.

Example: Figure 14 depicts an example of Black capturing the piece S^{15} by multiplication, denoted

`[`c^5`m4 * 3]`$\times S^{15}$`i4 (multiplication)`

If you think about it, there can't be a "huge" number of possible captures by multiplication. The reason why is the *multiplication capture value constraint*,

$$\mathtt{value}_1 * \mathtt{distance} = \mathtt{value}_2, \tag{1}$$

along with the the fact that both sides have only about 20 or so values to begin with means we can list (with the aid of a computer to keep track of the terms) all the possible cases in which equation (1) holds. Some captures might be ruled out due to the parity conditions in Lemmas 3 and 4, but at least we will have an upper bound which is hopefully stll pretty close. These upper bounds are described below.

There are at most 96 ways for White to capture Black by multiplication. Some examples that are not capture by numbering are listed below:

$$[S^{45} * 5] \times s^{225}, \quad [S^{45} * 2] \times t^{90}, \quad [S_1^{25} * 9] \times s^{225},$$
$$[S_2^{25} * 9] \times s^{225}, \quad [S_1^{25} * 4] \times t^{100}, \quad [S_2^{25} * 4] \times t^{100}, \ldots$$

There are at most 61 ways for Black to capture White by multiplication. Some examples that are not capture by numbering are listed below:

$$[s^{16} * 4] \times C^{64}, \quad [t^{12} * 6] \times T^{72}, \quad [c_1^9 * 9] \times T^{81},$$
$$[c_2^9 * 9] \times T^{81}, \quad [c_1^9 * 9] \times S^{81}, \quad [c_2^9 * 9] \times S^{81},$$
$$[c^7 * 13] \times P^{91}, \quad [c^7 * 7] \times T^{49}, \quad [c^3 * 14] \times T^{42}, \quad [c^3 * 12] \times C^{36}, \ldots$$

■49	■121					■225	■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81	▲90	▲100	n
		●3	●5	●7	●9			m
								l
								k
								j
			□15					i
								h
								g
								f
								e
		○8	○6	○4	○2			d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169						□25	a
1	2	3	4	5	6	7	8	

Figure 14: Rithmomachia position before a Black capture by multiplication: c^5 takes S^{15}, or $c^5 * 3 \times S^{15}$

Question 7. *Suppose on the very first move, White/Even plays* T^{72}`c2e2`*. Can you find a move by Black/Odd that will enable the capture of that White* △*?*

Question 8. *What is the exact number of ways for Black to capture White by multiplication? Same question for White to capture Black by multiplication.*

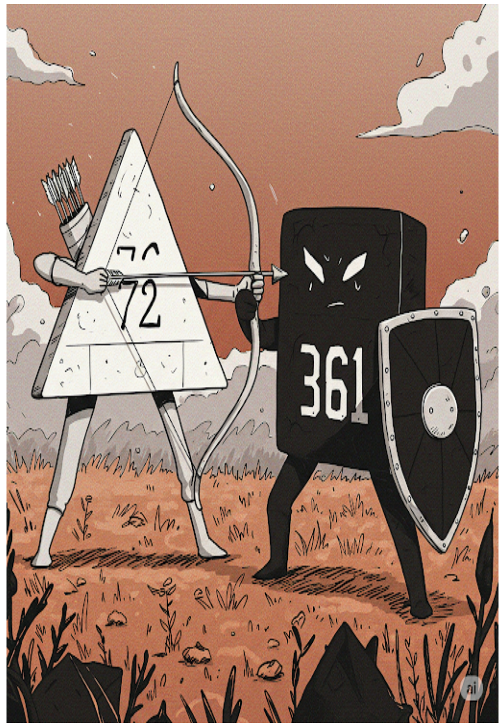


Figure 15: rithmomachia pieces can get intense. (Thanks gemini:-)

4.5.5 Division

Similar to multiplication: A piece of value $d{\cdot}v$ can capture an enemy piece of value v *if* the number of vacant coordinates along an unoccupied straight line (vertical, horizontal) drawn between these two pieces is exactly $d > 0$. Like capture by multiplication, this capture by division boils down to capture by numbering if $d = 1$.

Capturing by division notation

A capture by division could be written something like this:

`[(attacking piece1+value:coord.)÷ distance))×(captured piece2+value2:coord.).`

However, often the context is clear and we simply write

`[(attacking piece1+value1)÷ distance]x(captured piece2+value2).`

Example: On White's turn, the white square S^{15} can capture c^5 by division (÷) in Figure 14. In coordinate and algebraic notation:

`[S15:(3,8)÷ 3] × c5:(3,12) (division)` or $[S^{15}i4 \div 3] \times c^5m4$.

There are symmetry considerations for this capture as well.

Lemma 5. *(multiplication/division symmetry principle) A White piece can capture a Black piece by division if and only if that Black piece can capture that White piece by multiplication.*

Looking at the counts for capturing by multiplication, the multiplication/division symmetry principle (Lemma 5 above) tells us:

- there are at most 96 ways for Black to capture White by division, and
- there are at most 61 ways for White to capture Black by division.

Question 9. *The game state displayed in the Rithmomachia diagram*

s^{49}	s^{121}					s^{225}	s^{361}
s^{28}	s^{66}	t^{36}	t^{30}	t^{56}	t^{64}	s^{120}	p^{190}
t^{16}	t^{12}	c^{9}	c^{25}	c^{49}	c^{81}	t^{90}	t^{100}
		c^{3}	c^{5}	c^{7}	c^{9}		
					T^{9}		
		C^{8}	C^{6}	C^{4}	C^{2}		
T^{81}	T^{72}	C^{64}	C^{36}	C^{16}	C^{4}	T^{6}	
S^{153}	P^{91}	T^{49}	T^{42}	T^{20}	T^{25}	S^{45}	S^{15}
S^{289}	S^{169}					S^{81}	S^{25}

has the curious property that a single piece can capture another by three of the six different methods. Can you name them?

○ $m \cdot x$	$\leftarrow \ldots$	m	$\ldots \rightarrow$	■ x

Table: Division: White piece of value mx capturing from a distance a Black piece of value x

Question 10. *Suppose on the very first move, White/Even played* T^{72}`c2e2` *and then Black/Odd played* c^{9}`m6l6`*, but no captures. Can you find White's (second) move that will enable the capture of a Black* ▲*?*

4.5.6 Siege

We recall the definition from [DG21]: If an enemy piece is surrounded by opposing pieces in such a way that the enemy piece has no legal moves then the "besieged" enemy piece may be captured. For example, this capture can take place by an edge of the board, with the enemy piece partially surrounded by the enemy, or away from the edge, with the piece entirely surrounded by enemy pieces.

Figure 16 depicts an example of White capturing the piece c^3 by siege.

Capturing by siege notation

A capture by siege could be written something like this: if `attacking piece1` is the last piece to move to help surround and block `captured piece2` then

$$\texttt{(attacking piece1+value:coord.)}\cap\texttt{(captured piece2+value2:coord.)}.$$

However, often the context is clear and we simply write

$$\texttt{(attacking piece1+value1)}\cap\ \texttt{(captured piece2+value2)}.$$

Example: If White just moved his circle C^8 to $d3$ then this notation becomes

$$C^8 d3 \cap c^3 c3\ \texttt{(siege)}$$

4.5.7 Captures of/with the pyramid

The pyramid piece can be captured using its total value. In addition, any one of its component sub-pieces can be captured. In case only a sub-piece of the pyramid is captured, only that subpiece is removed from the board, never to return, and the value of that sub-piece is deducted from that of the pyramid piece. In the case the pyramid is captured using its full value, the entire piece (and all its component sub-pieces) must be removed from the board.

Example 6. *For the original pyramid piece* p^{190} *(see §4.4.3 above), Figure 17 depicts a game state in which the component sub-piece* s^{36}*, the black square sub-piece of* p *with value* 36*, can capture the white circle* C^{36}*.*

■49	■121					■225	■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81	▲90	▲100	n
			●5	●7	●9			m
								l
								k
								j
								i
								h
								g
								f
								e
		○8	○6	○4	○2			d
△81	△72	●3	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Figure 16: Rithmomachia position before a capture by siege, c^3 is surrounded by White and can't move.

Question 11. *What other captures (for both sides) can you find in the study depicted in Figure 17?*

The pyramid can capture an enemy piece, using its total value. Alternatively, the pyramid can capture an enemy piece including a component sub-piece of the enemy's pyramid, using the value of any one of its component sub-pieces.

Example 7. *If $P = S^{91}$ was against the edge of the board and surrounded* *on the three other sides by c^3, c^7, and c^{81}, then Black can capture P (as a whole) by siege. However, it cannot capture P by addition.*

■49	■121					■225	■361	*p*
■28	■66	▲36	▲30	▲56	▲64	■120		*o*
▲16	▲12		●25	●49	●81	▲90	▲100	*n*
		●3	●5	●7	●9			*m*
			●9					*l*
								k
								j
			◇ P^{91}					*i*
								h
								g
			◆ p^{190}					*f*
								e
		○8		○4	○2			*d*
△81	△72	○64	○36	○16	○4	△6	△9	*c*
□153		△49	△42	△20	△25	□45	□15	*b*
□289	□169					□81	□25	*a*
1	2	3	4	5	6	7	8	

,

Figure 17: A rithmomachia position before the triple capture using component subpieces of the pyramid: $p^{190}f4 \times S^{16}i4$, $p^{190}f4 \times S^{25}i4$, $p^{190}f4 \times S^{36}i4$. This chain capure is an example of a series of capture by numberings.

Here is a partial game involving a pyramid capture:

	Even moves	*captures*	*Odd moves*	*captures*
1	T^9`c8e8`		t^{12}`n2l2`	

This last move by Black makes a threat that White ignores.

	Even moves	*captures*	*Odd moves*	*captures*
2	T^6`c7e7`		t^{12}`l2j2`	$t^{12} \times T^{72}$
3	T^9`e8g8`		t^{12}`j2h2`	

This last move by Black makes a threat that White also ignores.

	Even moves	*captures*	*Odd moves*	*captures*
4	S^{15}`b8e8`		t^{12}`h2f2`	$t^{12} \times S^{36}$

This last partured piece S^{36} was a subpiece of the pyramid P^{91}. That pyramid has had it's value deducted, making it now

$$P^{55} = S^{25} + S^{16} + S^9 + S^4 + S^1.$$

Example 8. *A slaughterfest of captures awaits the curious player in the example study depicted in Figure 18.*

Question 12. *Can you find all captures in Figure 18, if*

(a) it is White's turn, and

(b) it is Black's turn?

4.6 How a game is won

As mentioned in §4.1, if it's a player's turn yet that player has no legal move then the player loses that rithmomachia game. There are several other ways to win a game of rithmomachia, broken down below into two distinct types.

● 7							△ 42	*p*
								o
			■ 361		△ 72	▲ 90		*n*
								m
								l
			□ 289	■ 49				*k*
								j
								i
				□ 169			■ 120	*h*
								g
								f
		○ 4						*e*
	○ 4	● 7	○ 16					*d*
		○ 2	● 9			△ 9		*c*
								b
◇ P^{91}						● 9		*a*
1	2	3	4	5	6	7	8	

Figure 18: All types of captures are illustrated in this rithmomachia study. Have fun finding them all!

How to Win a Game

A player loses when it is their turn and no legal move can be made, or when

- *common*: The number and values of the captured pieces satisfy certain conditions.
- *proper*: A certain type of configuration of at least 3 enemy pieces is arranged in friendly territory.

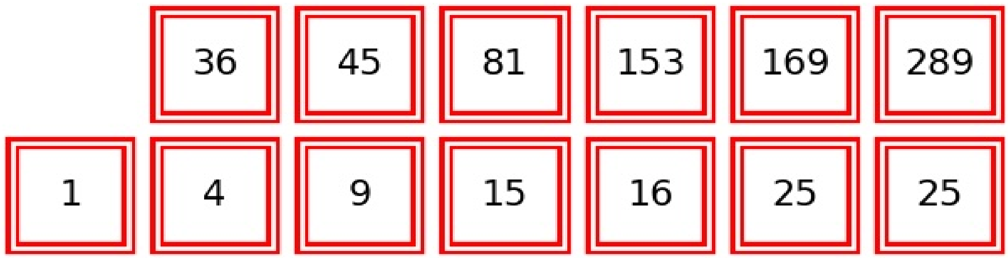


Figure 19: Rithmomachia White square pieces

One type of win is called *common* (or *minor*) and the other type of win is called *proper* (or *major*). As explained below, some types of victory discussed here *require* White and Black to agree upon a certain conventions before the game.

4.6.1 Common

As mentioned, in addition to common, this is also called a minor victory, because it is somewhat easier to manage.

- Assume that White and Black previously agreed upon a value for the number of captured pieces, say, N_0. As an example, de Boissière gives $N_0 = 4$ (see page 204 of [Ri46]). Another source suggests a larger proportion of the pieces, say 50% (so $N_0 = 12$).

  The first player to capture N_0 of the enemy's pieces (irregardless of their values) is declared the *winner by body* (sometimes called a *de Bonis victory*).

- Assume that White and Black previously agreed upon a number for the total value of captured pieces, say, N_1. For example, de Boissière gives $N_1 = 100$ (see page 205 of [Ri46]). Another source suggests a larger proportion of the total value, say 50%. Recall from Figures 6 and 5, the total value of the pieces depends on whether they are Odd/Black or Even/White pieces, but in any case that total is in the range 1300-to-1800.

  The first player to capture enemy's pieces whose total equals or exceeds N_1 is declared the *winner by goods* (also called a *de Lite victory*).

Another alternative is for White and Black to agree that such a victory only occurs is N_1 is attained exactly by a combination of captured enemy piece values.

- Assume that White and Black previously agreed upon two numbers, N_2 and N_3, for the total value of a number of captured pieces. As an example, de Boissière gives $N_2 = 4$ and $N_3 = 100$ (see page 205 of [Ri46]).

  The first player to capture at least N_2 enemy's pieces whose total equals or exceeds N_3 is declared the *winner by law-suit*. The winner can capture more than N_2 pieces but of them, there are N_2 captured enemy pieces whose total is at least N_3.

- This type of victory is similar to the law-suit, but the total value must be attained exactly. The first player to capture N_2 enemy's pieces whose total equals N_3 is declared the *winner by honor*. The winner can capture more than N_3 pieces but of them, there are N_2 captured enemy pieces whose total is exactly N_3.

- This type of victory is similar to the honor and the law-suit, but the total number of captures and the total value must both be attained exactly.

  We use the same notation. The first player to capture N_2 enemy's pieces (no more, no less) whose total equals N_3 is declared the *winner by honor and law-suit*. To win by this method, the player must capture exactly N_2 enemy pieces and their total must be exactly N_3, no more and no less.

These are all the common victories.

Common victories:

- **By Body:** First to capture an agreed number of enemy pieces (e.g., $N_0 = 4$).
- **By Goods:** First to capture enemy pieces whose total value equals or exceeds an agreed sum (e.g., $N_1 = 100$).
- **By Law-suit:** Capture $\geq N_2$ pieces whose total value is $\geq N_3$ (e.g., $N_2 = 4, N_3 = 100$).
- **By Honor:** Capture N_2 pieces whose total value is exactly N_3.

4.6.2 Proper

> ...this is called a proper victory with good reason. For just as those who make war win a victory after routing the enemy's battle-line and arranging everything in accordance with their own will, so too the harmony and arrangement of numbers ...
>
> Claude de Boissière ([Ri46], pages 206-207)

Each of these proper victories involve placing a fixed number of friendly pieces in the enemy territory such that the values of these pieces form eith an arithmetical or geometrical or harmonic progression. We will call any such progressions a *coordinating progression*. Such progressions were known to Pythagoras and covered in the book by Nicomachus [Ni00].

There are three types of proper wins (see [Ri46] pages 206-212, [Ne98], [DG21].

- *Small*,
- *Big*,
- *Great*.

Each of these win types require a certain (coordinating) arrangement of 3 or 4 pieces, the arrangement being a pattern of 3 or 4 values of pieces in enemy territory must all belong to either an *arithmetic* progression, a *geometric* progression, or a *harmonic* progression. Definitions and a examples of some of these can be found in Figure 20.

Since each side has only about 20 or so values amongest their pieces, there are lots of 3-term or 4-term arithmetic progressions which *aren't* formed from these values. We used the programming language Python to extract the 3-term arithmetic/geometric/harmonic progressions which *are* formed from three piece values. The algorithm used did not distinguish between pieces *themselves* but rather between the values of the pieces. Of course, the values of pieces can repeat (like Odd/Black's `c9`s and Even/White's `T6` and `C6`). These lower bounds are woven in the discussion below.

- An *arithmetical progression* is a sequence a, b, c satisfying $b - a = c - b$.

  - White has at least 11 such progressions:

$$\{(2,4,6),(2,9,16),(4,6,8),(4,20,36),(8,25,42),(8,36,64),$$
$$(9,45,81),(9,81,153),(15,20,25),(20,42,64),(49,169,289)\},$$

  - Black has at least 9 such progressions:

$$\{(3,5,7),(5,7,9),(7,16,25),(7,28,49),(7,64,121),$$
$$(12,56,100),(12,66,120),(16,36,56),(28,64,100)\}.$$

  Question 13. *Accounting for the repeated values can you determine the exact number of arithmetical progressions possible for White? For Black?*

- A *geometric progression* is a sequence a, b, c satisfying $b/a = c/b$.

  For example, White has at least 11 such progressions:

$$\{(2,4,8),(4,6,9),(4,8,16),(4,16,64),(9,15,25),(16,20,25),$$
$$(16,36,81),(25,45,81),(36,42,49),(64,72,81),(81,153,289)\}.$$

  Black has at least 10 such progressions:

$$\{(9,12,16),(9,30,100),(16,28,49),(16,36,81),(25,30,36),$$
$$(36,66,121),(36,90,225),(49,56,64),(64,120,225),(81,90,100)\}.$$

arithmetical			geometric			harmonic		
2	4	6	2	4	8	9	15	45
3	**5**	**7**	**100**	**190**	**361**	9	16	72

Figure 20: Example coordinating configurations for a *proper win*.

Question 14. *Accounting for the duplicate piece values can you determine the exact number of geometrical progressions possible for White? For Black?*

- The *harmonic progressions* a, b, c in each row of the harmonic table satisfy $1/a - 1/b = 1/b - 1/c$, so their reciprocals form an arithmetical sequence.

  White has 2 of them $(9, 15, 45)$ and $(9, 16, 72)$, while Black has none.

  Question 15. *Can you determine the smallest number of White must make in a game of rithmomachia to achieve a small proper win by harmonic progression with the sequence* $9, 15, 45$*?*

Offensive capabilities

Each player who wants to achieve such a small proper win will naturally want to have as many or more triples of pieces which form a coordinating progression. To win, the pieces in such an "offensive" triple (a *fireteam*, in the battlefield lingo) must all invade enemy territory. The White/Even player has 60 pairs of values which belong to at least one coordinating progression. For example, each of the pairs $(4, 6)$ and $(20, 25)$ belong to at least *two* coordinating progressions. The Black/Odd player has 57 pairs of values which belong to at least one coordinating progression. For example, each of the pairs $(5, 7)$ and $(28, 49)$ belong to at least *two* coordinating progressions. From this perspective, the two sides seem to have roughly equal offensive capabilities.

According to de Boissière (see [Ri46], page 206), such coordinating progressions of pieces are a metaphor for the cooperation and coordination of the corresponding fireteam on the battlefield.

Proper victories:

- **Small victory:** Place **three** friendly pieces in enemy territory whose values a, b, c form an arithmetic, geometric, or harmonic progression.
- **Big victory:** Four pieces in enemy territory; one subset of three forms a coordinating progression, and another subset of three forms another.
- **Great victory:** Among the four pieces, there are subsets of 3 that form arithmetic, geometric, and harmonic progressions.

In other words, a player wins by a great victory when they have a big win but, moreover, there exists among the four pieces in enemy territory,

- a subset of three of the four whose values form an arithmetic progression,
- a subset of three of the four whose values form a geometric progression, and
- a subset of three of the four whose values form a harmonic progression.

Remark 4. In some versions of rithmomachia, these three pieces must be arranged in a straight line (vertical, horizontal, or diagonal). Other versions of rithmomachia allow these pieces to be arranged in the shape of a right angled triangle. No such linear or geometric conditions exist for the de Boissière version here.

Example: Figure 21 depicts an example of White winning by geometrical progression. White has three pieces, at $i1$, $i4$, $i7$, in enemy territory with values $(2, 4, 8)$ in geometrical progression.
Example: The game in §5 provides an example of a small geometric (proper) victory of Even/White over Odd/Black.

Question 16. *Can you find an example of a final position that achieves a big victory winning position?*

Question 17. *Can you find an example of a final position that achieves a great victory winning position?*

■49	■121					■225	■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81	▲90	▲100	n
		●3	●5	●7	●9			m
								l
								k
								j
○2			○4			○8		i
								h
								g
								f
								e
			○6					d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Figure 21: A Rithmomachia position depicting a White win by geometrical harmony.

5 Two example games

A couple of example games are given next.

5.1 An arithmetical win

In this case, it's easy to describe this game. White has a plan to place three pieces in a coordinating progression. Black is moves more-or-less randomly

Figure 22: Yeah, rithmomachia ! (thanks chatGPT:-)

so has little chance of stopping this plan from being implemented.

White plays first. After the first few moves, the position of the board and moves were as follows. The last move played is underlined.

The first few turns

	Even moves	Odd moves
1	`C002d6e6`	`s121p2p5`

■49				■121		■225	■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81	▲90	▲100	n
		●3	●5	●7	●9			m
								l
								k
								j
								i
								h
								g
								f
					○2			e
		○8	○6	○4				d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Round 2 game state

	Even moves	captures	odd moves	captures
1.	`C002d6e6`		`s121p2p5`	
2.	`C002e6f6`		`c007m5l5`	

■49				■121		■225	■361	*p*
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	*o*
▲16	▲12	●9	●25	●49	●81	▲90	▲100	*n*
		●3	●5		●9			*m*
				●7				*l*
								k
								j
								i
								h
								g
					○2			*f*
								e
		○8	○6	○4				*d*
△81	△72	○64	○36	○16	○4	△6	△9	*c*
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	*b*
□289	□169					□81	□25	*a*
1	2	3	4	5	6	7	8	

Round 3 game state

	Even moves	odd moves
1	`C002d6e6`	`s121p2p5`
2	`C002e6f6`	`c007m5l5`
3	`C002f6g6`	<u>`c009m6m5`</u>

■49				■121		■225	■361	*p*
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	*o*
▲16	▲12	●9	●25	●49	●81	▲90	▲100	*n*
		●3	●5	●9				*m*
				●7				*l*
								k
								j
								i
								h
					○2			*g*
								f
								e
		○8	○6	○4				*d*
△81	△72	○64	○36	○16	○4	△6	△9	*c*
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	*b*
□289	□169					□81	□25	*a*
1	2	3	4	5	6	7	8	

Round 4 game state

	Even moves	captures	odd moves	captures
1	`C002d6e6`		`s121p2p5`	
2	`C002e6f6`		`c007m5l5`	
3	`C002f6g6`		`c009m6m5`	
4	`C002g6h6`		`s121p5p2`	

■49	■121					■225	■361	*p*
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	*o*
▲16	▲12	●9	●25	●49	●81	▲90	▲100	*n*
		●3	●5	●9				*m*
				●7				*l*
								k
								j
								i
					○2			*h*
								g
								f
								e
		○8	○6	○4				*d*
△81	△72	○64	○36	○16	○4	△6	△9	*c*
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	*b*
□289	□169					□81	□25	*a*
1	2	3	4	5	6	7	8	

White's 2-circle `C2` is finally in the desired position:

Round 5 game state

	Even moves	captures	odd moves	captures
1	C002d6e6		s121p2p5	
⋮	⋮		⋮	
4	C002g6h6		s121p5p2	
5	C002h6i6		t090n7l7	

■49	■121					■225	■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81		▲100	n
		●3	●5	●9				m
				●7		▲90		l
								k
								j
					○2			i
								h
								g
								f
								e
		○8	○6	○4				d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Round 6 game state

	Even moves	captures	odd moves	captures
1	`C002d6e6`		`s121p2p5`	
⋮	⋮	⋮	⋮	⋮
5	`C002h6i6`		`t090n7l7`	
6	`C004d5e5`		<u>`c005m4l4`</u>	

■49	■121					■225	■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81		▲100	n
		●3		●9				m
			●5	●7		▲90		l
								k
								j
					○2			i
								h
								g
								f
				○4				e
		○8	○6					d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Round 7 game state

	Even moves	captures	odd moves	captures
1	`C002d6e6`		`s121p2p5`	
⋮	⋮	⋮	⋮	⋮
6	`C004d5e5`		`c005m4l4`	
7	`C004e5f5`		`s225p7p4`	

■49	■121		■225				■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81		▲100	n
		●3		●9				m
			●5	●7		▲90		l
								k
								j
					○2			i
								h
								g
				○4				f
								e
		○8	○6					d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Round 8

	Even moves	captures	odd moves	captures
1	`C002d6e6`		`s121p2p5`	
⋮	⋮	⋮	⋮	⋮
7	`C004e5f5`		`s225p7p4`	
8	`C004f5g5`		`c009m5m4`	

■49	■121		■225				■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81		▲100	n
		●3	●9					m
			●5	●7		▲90		l
								k
								j
					○2			i
								h
				○4				g
								f
								e
		○8	○6					d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Round 9

	Even moves	even captures	odd moves	odd captures
1	`C002d6e6`		`s121p2p5`	
⋮	⋮	⋮	⋮	⋮
8	`C004f5g5`		`c009m5m4`	
9	`C004g5h5`		`c081n6n7`	

■49	■121		■225				■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49		●81	▲100	n
		●3	●9					m
			●5	●7		▲90		l
								k
								j
					○2			i
				○4				h
								g
								f
								e
		○8	○6					d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

White/Even's 4-circle `C4` is in enemy territory (in row i):

Round 10

	Even moves		odd moves	
1	`C002d6e6`		`s121p2p5`	
⋮	⋮	⋮	⋮	⋮
9	`C004g5h5`		`c081n6n7`	
10	`C004h5i5`		`c081n7n6`	

■ 49	■ 121		■ 225				■ 361	p
■ 28	■ 66	▲ 36	▲ 30	▲ 56	▲ 64	■ 120	◆ p^{190}	o
▲ 16	▲ 12	● 9	● 25	● 49	● 81		▲ 100	n
		● 3	● 9					m
			● 5	● 7		▲ 90		l
								k
								j
				○ 4	○ 2			i
								h
								g
								f
								e
		○ 8	○ 6					d
△ 81	△ 72	○ 64	○ 36	○ 16	○ 4	△ 6	△ 9	c
□ 153	◇ P^{91}	△ 49	△ 42	△ 20	△ 25	□ 45	□ 15	b
□ 289	□ 169					□ 81	□ 25	a
1	2	3	4	5	6	7	8	

Round 11 game state

	Even moves	even captures	odd moves	odd captures
1	`C002d6e6`		`s121p2p5`	
⋮	⋮	⋮	⋮	⋮
10	`C004h5i5`		`c081n7n6`	
11	`C006d4e4`		`t090l7j7`	

■49	■121		■225				■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81		▲100	n
		●3	●9					m
			●5	●7				l
								k
						▲90		j
				○4	○2			i
								h
								g
								f
			○6					e
		○8						d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Presumably, White's strategy is clear now: to win by (small proper) arithmetical progression.

Round 12 game state

	Even moves	even captures	odd moves	odd captures
1	`C002d6e6`		`s121p2p5`	
⋮	⋮	⋮	⋮	⋮
11	`C006d4e4`		`t090l7j7`	
12	`C006e4f4`		`c003m3l3`	

■49	■121		■225				■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81		▲100	n
			●9					m
		●3	●5	●7				l
								k
						▲90		j
				○4	○2			i
								h
								g
			○6					f
								e
		○8						d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Round 13 game state

	Even moves	even captures	odd moves	odd captures
1	`C002d6e6`		`s121p2p5`	
⋮	⋮	⋮	⋮	⋮
12	`C006e4f4`		`c003m3l3`	
13	`C006f4g4`		`c009m4m3`	

■49	■121		■225				■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81		▲100	n
		●9						m
		●3	●5	●7				l
								k
						▲90		j
				○4	○2			i
								h
			○6					g
								f
								e
		○8						d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Round 14 game state

	Even moves	odd moves
1	`C002d6e6`	`s121p2p5`
⋮	⋮	⋮
12	`C006e4f4`	`c003m3l3`
13	`C006f4g4`	`c009m4m3`
14	`C006g4h4`	<u>`c007l5k5`</u>

■49	■121		■225				■361	p
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	o
▲16	▲12	●9	●25	●49	●81		▲100	n
		●9						m
		●3	●5					l
				●7				k
						▲90		j
				○4	○2			i
			○6					h
								g
								f
								e
		○8						d
△81	△72	○64	○36	○16	○4	△6	△9	c
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	b
□289	□169					□81	□25	a
1	2	3	4	5	6	7	8	

Round 15: The strategy for White/Even is very simple: march three pieces to enemy territory. If their values form a coordinating progression then White/Even can declare a small proper victory over Black/Odd.

After 15 moves, finally White/Even now has an arithmetical progression on row j, which is in enemy territory.

	Even moves	even captures	Odd moves	odd captures
1	`C002d6e6`		`s121p2p5`	
2	`C002e6f6`		`c007m5l5`	
3	`C002f6g6`		`c009m6m5`	
4	`C002g6h6`		`s121p5p2`	
5	`C002h6i6`		`t090n7l7`	
6	`C004d5e5`		`c005m4l4`	
7	`C004e5f5`		`s225p7p4`	
8	`C004f5g5`		`c009m5m4`	
9	`C004g5h5`		`c081n6n7`	
10	`C004h5i5`		`c081n7n6`	
11	`C006d4e4`		`t090l7j7`	
12	`C006e4f4`		`c003m3l3`	
13	`C006f4g4`		`c009m4m3`	
14	`C006g4h4`		`c007l5k5`	
15	`C006h4i4`		1-0	

Here 1-0 means the first player (White or Even) has scored a win (so earns the 1) over Black or Odd (who sadly scores 0).

game state:

■49	■121		■225				■361	*p*
■28	■66	▲36	▲30	▲56	▲64	■120	◆p^{190}	*o*
▲16	▲12	●9	●25	●49	●81		▲100	*n*
		●9						*m*
		●3	●5					*l*
				●7				*k*
						▲90		*j*
			○6	○4	○2			*i*
								h
								g
								f
								e
		○8						*d*
△81	△72	○64	○36	○16	○4	△6	△9	*c*
□153	◇P^{91}	△49	△42	△20	△25	□45	□15	*b*
□289	□169					□81	□25	*a*
1	2	3	4	5	6	7	8	

Figure 23: Rithmomachia Black/Odd triangles pieces in a line

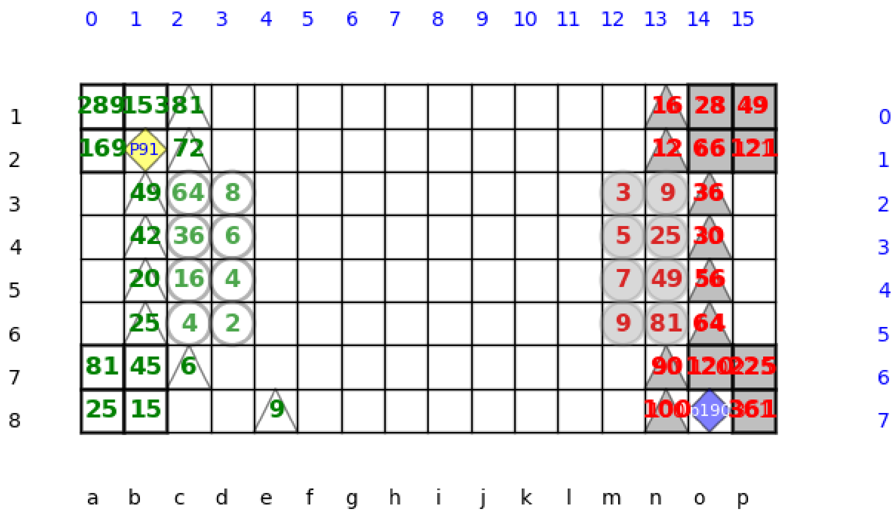
5.2 A harmonic win

The moves of the second game analyzed here is recorded as follows:

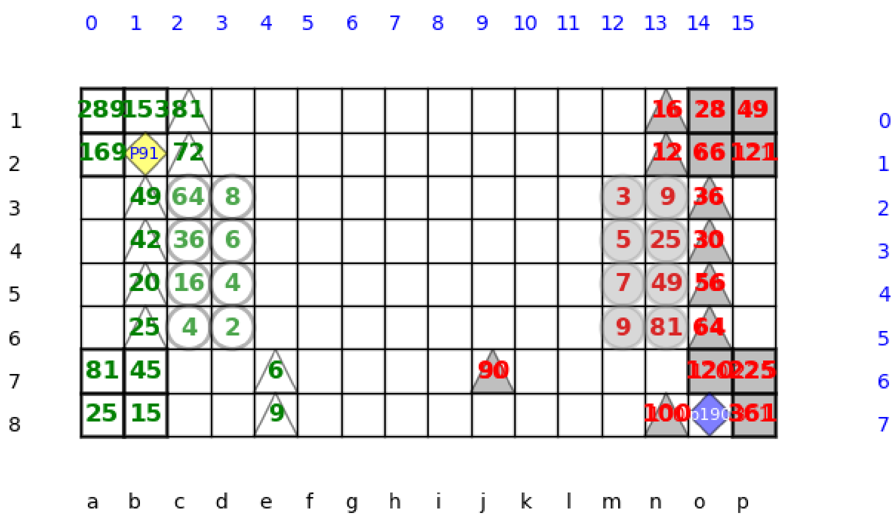
	Even moves	*captures*	*Odd moves*	*captures*
1	T^9`c8e8`		t^{90}`n7l7`	
2	T^6`c7e7`		t^{90}`l7j7`	
3	T^9`e8g8`		s^{120}`o7l7`	
4	S^{15}`b8e8`		s^{225}`p7m7`	
5	T^9`g8i8`		s^{121}`p2p5`	
6	T^6`e7e5`		t^{16}`n1l1`	
7	S^{15}`e8h8`		t^{16}`l1j1`	
8	T^6`e5g5`		s^{28}`o1l1`	
9	S^{15}`h8h5`		s^{49}`p1m1`	
10	S^{15}`h5k5`		t^{90}`j7h7`	
11	S^{45}`b7e7`	$S^{45} \times t^{90}$	s^{120}`l7i7`	
12	S^{45}`e7h7`		s^{225}`m7j7`	
13	S^{45}`h7h4`		s^{120}`i7f7`	
14	S^{45}`h4k4`		$1-0$	

The Python program for playing rithmomachia was revised since the game in the previous section. Now Black's strategy is more in line with defending against a possible White invasion.

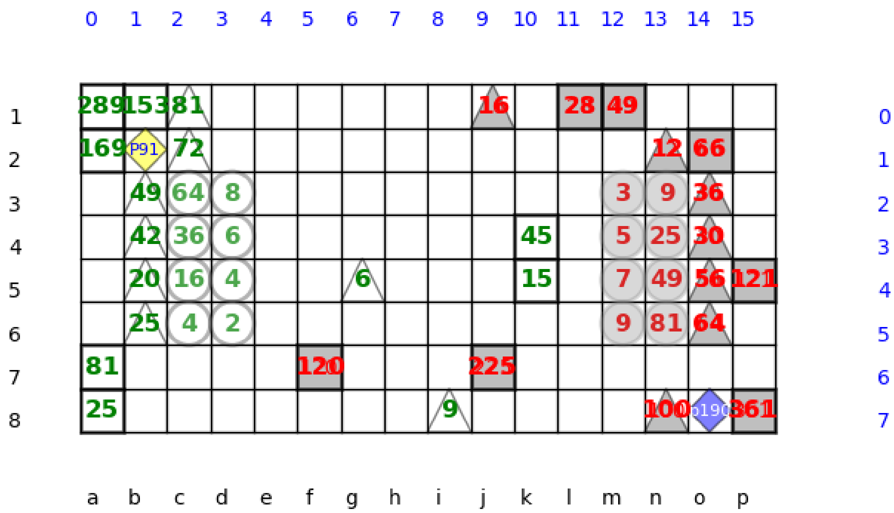
After White's first move, the position is depicted here:



After the second Black move, the position is depicted here:



Imagine, in this position, that the triangle T^6 was moved to the side, so there is a clear shot from T^{45} to t^{90}. You can see that White can't move the S^{45} piece ahead in the 225 file without fear of being captured (by division) by t^{90} on the next move. In some sense, Black's blockage using t^{90} has successfully stymied White's plans for now. However, White evades the blockage by making a few moves to switch files and invade via the 42 file. The final position is here:



6 Last but not least

Rithmomachia is a board game emphasizing pattern recognition, logical thinking, and numerical relationships between the pieces. It has an interesting history, and presents us with a pleasant diversion involving logic, strategy, twinged with philosophical ideas.

6.1 Open questions

A few open questions remain.

Question 18. *Assuming best play, does one side had an advantage over the other?*

There are arguments on both sides:

- We know that Odd/Black has higher total piece value (1752) than Even/White (1312), so Odd/Back has the advantage from this point of view.
- On the other hand,
  (a) there are at most 118 ways for White to capture Black by addition, and

(b) there are at most 141 ways for White to capture Black by subtraction.

Compare that with:

(c) there are at most 82 ways for Black to capture White by addition, and

(d) there are at most 119 ways for Black to capture White by subtraction.

So it seems Even/White has the advantage from this perspective.

Defense

As we've seen in the example games, a proper win can be quickly achieved by invading the enemy territory with a fireteam. To counter White's infiltration strategy, Black can perform this checklist:

- scan for two White pieces in Black territory that are part of a potential coordinating sequence.
- Identify if a third White piece can be moved into Black territory to complete this sequence.
- If such a threat exists, Black must either (a) capture one of White's two pieces already in the formation, (b) block the path of the incoming third White piece.

Easier said than done!

As far as we know, this next one is also an open question.

Question 19. *Will Odd/Black always win with perfect play from both sides in rithmomachia?*

6.2 Summary of Rithmomachia Rules and Setup

- **Board:** The game is played on an 8-by-16 rectangular board.

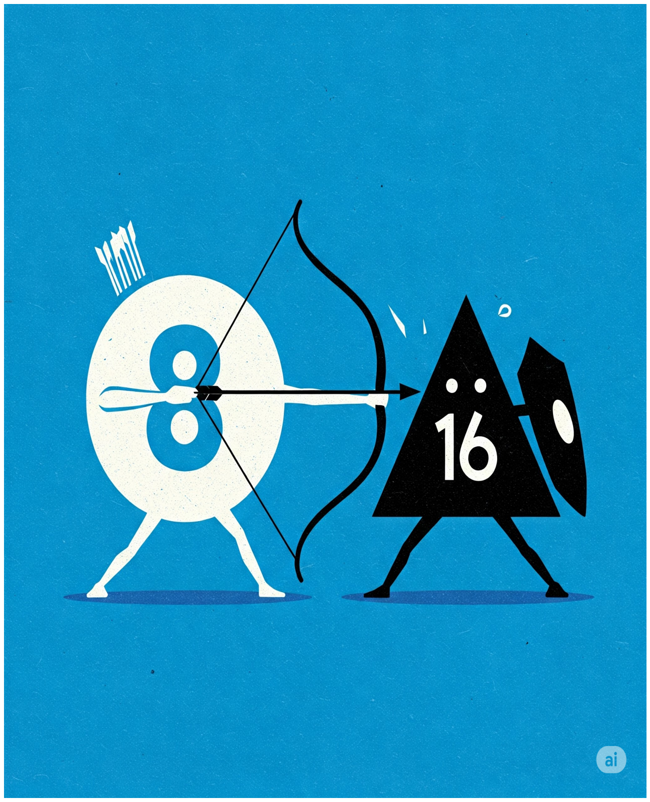


Figure 24: 𝔯𝔦𝔱𝔥𝔪𝔬𝔪𝔞𝔠𝔥𝔦𝔞 pieces can get mad too. Watch out especially for that hot head ○8! (Thanks gemini:-)

- **Players and Pieces:** Each player controls 24 pieces (white/even and black/odd), comprising: 8 *circles*, 8 *triangles*, and 8 *squares*. One of these squares is a special multi-valued piece called a *pyramid*.
- **Movement:**
  - Circles move orthogonally one.
  - Triangles move exactly two spaces orthogonally.
  - Squares and pyramids move exactly three spaces orthogonally.
  - No piece may jump over another; movement must be unobstructed.
- **Capturing:**
  - *capture by numbering:* The attacking piece is one move away.

  - *capture by addition/subtraction:* One move away for each of the attacking pieces.
  - *capture by multiplication/division:* Attacking piece at distance d, if its value equals d (divided by/times) the enemy's value.
  - *Siege capture:* An enemy piece is flanked on all sides by friendly pieces.

- **Victory Conditions:**
  - **Proper victory:** Three of your pieces in arithmetic, geometric, or harmonic progression on the opponent's side of the board.
  - **Common victory:** Some pre-agreed proportion of your opponents pieces (by count or by total value).

- **Philosophical and Mathematical Themes:**
  - Game emphasizes harmony, proportion, and the balance of opposites (odd/even).
  - Influenced by Pythagorean number mysticism and Boethian arithmetic.
  - Intended as a didactic tool for exploring numerical relationships and moral virtues.

6.3 Play a game!

If you're still curious, then play a game. A board is simply an 8×16 grid of squares – easy to make from cardboard and even two chess/checker boards side-by-side will work. Pieces aren't hard to make with cardboard, scissors, and a couple of different colored pens/crayons.

Hopefully, you'll have a little fun bringing this forgotten game back to life. If finding a human opponent is challenging, consider exploring computer versions, though (*caveat emptor*) be mindful that rule variations may exist.

You can even reach out to the author for a free copy of their Python/`SageMath` program to get started. The good news is that it's free, while the bad news is that there is a learning curve to installing and using the mathematical software program `SageMath` that my program currently relies on. The example in §5 was played using that program.

Answer to Question 1: $u = (2a+1)^2$, $v = (2b+1)^2$, $w = (2c+1)^2$, $x = (2d+1)^2$.

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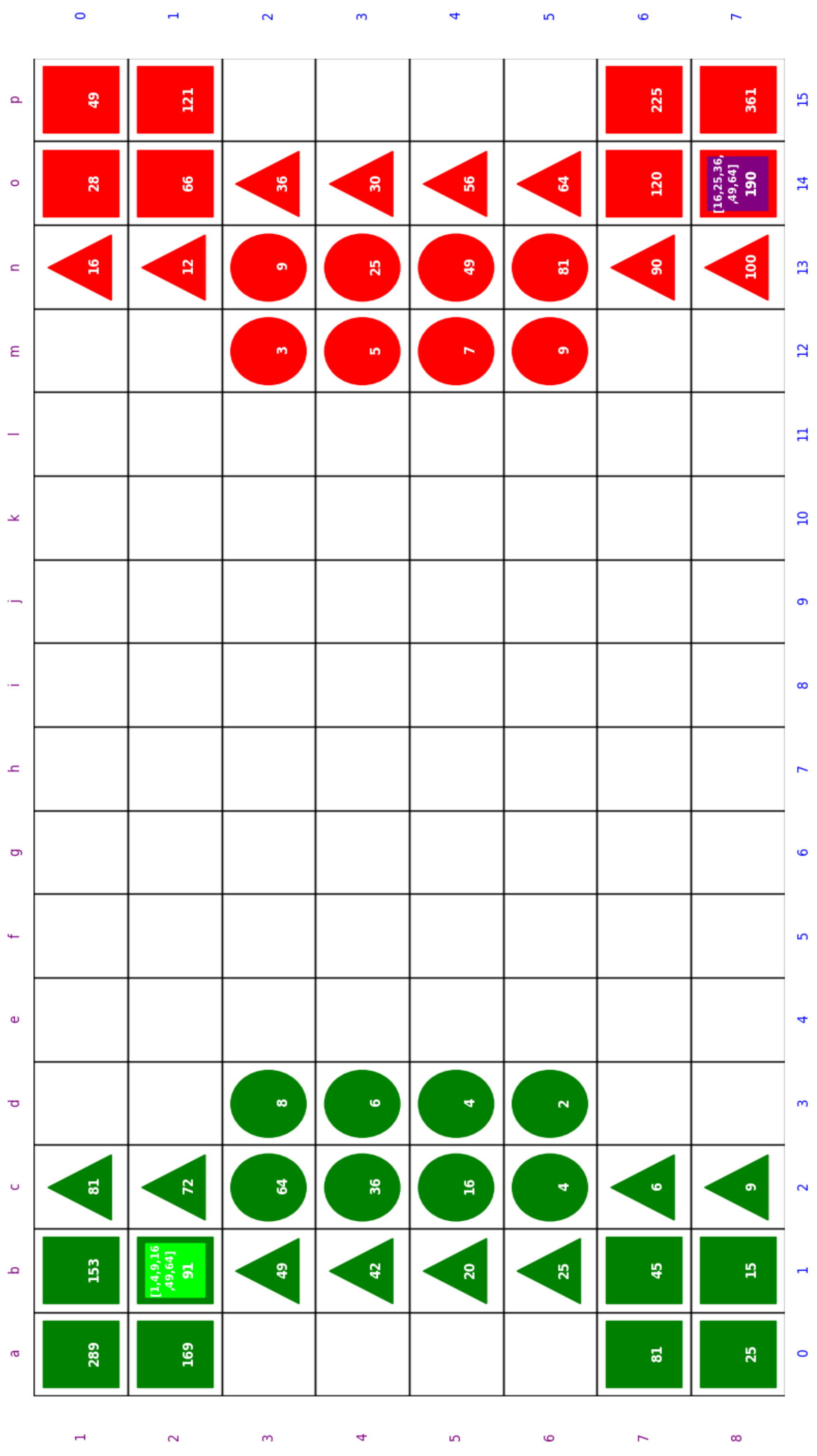

Figure 25: The rithmomachia board.